Difference posets and orthoalgebras [1]

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Abstract

We show that (and how) difference posets are isomorphic to orthoalgebras. As a corollary, we find a characterization of orthomodular lattices and Boolean algebras in terms of a difference operation. The results are relevant to the logico-algebraic foundation of quantum mechanics. (It should be noted that the manuscript form of this paper has circulated among interested readers for a longer time. As a result, the contents of this paper has been referred to in numerous papers, notably in the papers [7, 4, 11].)

1 Introduction

When we deal with a complemented poset we usually derive the difference operation (of comparable pairs) as a secondary notion. Taking the difference operation as primitive and imposing certain natural conditions on it, we obtain a *difference poset*. As it turns out, difference posets enjoy conceptual merits which might be utilized in quantum axiomatics and in the study of observables — see [11], see also [9], [12]). In this note we point out an explicit relationship of difference posets with structures already established in the foundation of quantum mechanics — with the *orthoalgebras* (see [6], [8], [10], [13], etc.). As a main result (Th. 2.11), we find a description of an orthoalgebra in terms of a difference operation on a poset. Using this description, we further obtain a characterization of orthomodular posets, orthomodular lattices and Boolean algebras in a difference poset setup.

2 Notions and results

Let us start with a basic notion of this paper (see also [11]).

Definition 2.1 : *Let $(S, \leq)$ be a partially ordered set and let S possess a least and a greatest element, 0, 1. Let $\ominus$ be a partial binary operation on S such that*

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$b \ominus a$ is defined if and only if $a \leq b$. Then $(S, 0, 1, \leq, \ominus)$ is called a difference poset *(a* DP*) if the following conditions are satisfied:*

(dp1) *For any $a \in S$, $a \ominus 0 = a$.*

(dp2) *If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.*

In what follows, we shall often use S instead of $(S, 0, 1, \leq, \ominus)$ if there is no danger of misunderstanding. Typical examples of DPs are as follows.

Example 2.2 : Let V be an ordered linear space and let f be a positive element of V. Put $V_f = \{g \in V \mid 0 \leq g \leq f\}$. Then V_f becomes a DP. In particular, the interval $[0, 1]$ of real numbers is a DP.

Example 2.3 : Let X be a set and let $\exp X$ denote the set of all subsets of X. Let Δ be a subset of $\exp X$ that contains X and that is closed under the formation of the set-theoretic difference of the sets which are in the inclusion relation. Then Δ with $\leq$ being the inclusion relation and $\ominus$ being the set-theoretic difference forms a DP.

Example 2.4 : Suppose that $(S, 0, 1, \leq, ')$ is an orthomodular poset (see e.g. [12]). If we put, for $a, b \in S$ such that $a \leq b$, $b \ominus a = b \wedge a'$, then S becomes a DP.

Example 2.5 : Let F be a collection of functions $f : X \to [0, 1]$ which fulfils the following properties:

1. the constant unit function $\mathbf{1}$ belongs to F,
2. if $f, g, \in F$, $f \leq g$, then $g - f \in F$.

Then F with $\ominus$ defined as the pointwise difference of functions is a DP. The latter DP plays an essential role in the quantum axiomatics based on fuzziness (see e.g. [2], [3], [5], etc.).

Definition 2.6 : *Let S, T be two DPs. A* DP morphism *is a mapping $\alpha : S \to T$ such that (i) $\alpha(1) = 1$, (ii) if $a, b \in S$ with $a \leq b$, then $\alpha(a) \leq \alpha(b)$ and $\alpha(b \ominus a) = \alpha(b) \ominus \alpha(a)$.*

As one can check easily, the class of DPs with the DP morphisms forms a category (see e.g. [1] for the category theory terminology). We shall denote the latter category by $\mathscr{DP}$.

Our intention in this note is to find a link of DPs with orthoalgebras. Let us first recall the definition of an orthoalgebra (see e.g. [8]).

Definition 2.7 : *Let K be a set containing two distinguished elements 0, 1 and let K be endowed with a partially defined binary operation $\oplus$ which satisfies the following four conditions ($a, b, c \in K$):*

(oa1) *If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.*

(oa2) *If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ is defined, $(a \oplus b) \oplus c$ is defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.*

(oa3) *For every $a \in K$ there exists a unique $b \in K$ such that $a \oplus b$ is defined and $a \oplus b = 1$.*

(oa4) *If $a \oplus a$ is defined, then $a = 0$.*

Then $(K, 0, 1, \oplus)$ is called an orthoalgebra *(an* OA*).*

In what follows, we shall often use K instead of $(K, 0, 1, \oplus)$ if there is no danger of misunderstanding. It should be noted that orthoalgebras have been found a useful tool in the pursuit of quantum mechanical constructs (see e.g. [6], [13], etc.).

Definition 2.8 : *Let K, L be two orthoalgebras. Then a mapping $\beta : K \to L$ is called an* OA morphism *if (i) $\beta(1) = 1$, (ii) if $a, b \in K$ and $a \oplus b$ exists, then $\beta(a) \oplus \beta(b)$ exists in L and $\beta(a \oplus b) = \beta(a) \oplus \beta(b)$.*

It can be shown easily that OAs with OA morphisms form a category. Let us denote this category by $\mathscr{OA}$.

In what follows, we shall be interested in the connection of DPs and OAs. The point of departure is the following observation.

Proposition 2.9 : *Let $(K, 0, 1, \oplus)$ be an orthoalgebra. Let us define a relation, $\leq$, and a partial binary operation, $\ominus$, in K as follows: For $a, b \in K$, we set $a \leq b$ if and only if there is a $c \in K$ such that $b = a \oplus c$, and in the latter case we put $b \ominus a = c$. Then $(K, 0, 1, \leq, \ominus)$ is a difference poset.*

Proof. One first has to check that the above defined relation $\leq$ is indeed a partial ordering making 0 and 1 a least and a greatest element. This has been done in [8]. (In fact, only the antisymmetry of $\leq$ needs to be verified, the other properties are obvious. Suppose that $a \leq b$ and $b \leq a$. Then there are elements c, d such that $b = a \oplus c$, $a = b \oplus d$. We have $a = (a \oplus c) \oplus d = a \oplus (c \oplus d) = a \oplus (c \oplus d) \oplus (c \oplus d)$. By (oa4), $c \oplus d = 0$. Thus, $c = d = 0$ and therefore $a = b$.)

Let us now verify the axioms of a DP for $(K, 0, 1, \leq, \ominus)$. Suppose that $a \leq b$. Thus, $b = a \oplus c$ and therefore $b \ominus a$ is defined and $b \ominus a = c$. Let $a \in K$. We have to prove that $a \ominus 0 = a$. This is equivalent to proving that $a = a \oplus 0$. To this end, observe first that $1 = 1 \oplus 0$. (Indeed, according to (oa3), there is a $b \in K$ such that $1 = 1\oplus b = (1\oplus b)\oplus b = 1\oplus(b\oplus b)$. By (oa2), $b\oplus b$ is defined and, by (oa4), $b = 0$.) Let $c \in K$ satisfy $c\oplus a = 1$. Then $c\oplus(a\oplus 0) = (c\oplus a)\oplus 0 = 1\oplus 0 = 1$ and, by (oa3), $a\oplus 0 = a$. Further, let us suppose that $a \leq b \leq c$. Thus, $b = a\oplus d$ and $c = b\oplus e$. This means that $c = a\oplus d\oplus e$. It follows that $(c\ominus a)\ominus(c\ominus b) = (d\oplus e)\ominus e = d = b\ominus a$ and the proof is complete. □

Theorem 2.10 : *The category $\mathscr{OA}$ is a full subcategory of $\mathscr{DP}$.*

Proof. By Prop. 2.9, every OA can be naturally viewed as a DP. Suppose that K, L are OAs and suppose further that $\beta : K \to L$ is an OA morphism. Then, obviously, $\beta(1) = 1$. If $a \leq b$, we have $a\oplus c = b$ for some $c \in K$. Thus, $\beta(a)\oplus\beta(c) = \beta(b)$ and therefore $\beta(a) \leq \beta(b)$. Moreover, the equality $\beta(a)\oplus\beta(c) = \beta(b)$ reads $\beta(c) = \beta(b\ominus a) = \beta(b)\ominus\beta(a)$. Thus, β is a DP morphism for K, L if K, L are understood as DPs. On the other hand, suppose that K, L are understood as DPs and $\alpha : K \to L$ be a DP morphism. Obviously, $\alpha(1) = 1$. If $a\oplus b = c$, then $a = c\ominus b$. It follows that $\alpha(a) = \alpha(c)\ominus\alpha(b)$ and therefore $\alpha(c) = \alpha(a)\oplus\alpha(b)$. Thus, α is an OA morphism and this completes the proof. □

A natural question now arises of which DPs can be given an orthoalgebra structure. Let K be a DP. Let us say that K is *regular* if it satisfies the following condition: If $a \in K$ and $a \leq 1\ominus a$, then $a = 0$. Let us denote by $\mathscr{DP}_{reg}$ the category of regular DPs. (Prior to the next result, let us observe that a DP does not have to be regular. In fact, none of the examples V_f (Ex. 2.2) is a regular DP.)

Theorem 2.11 : *Let $(K, 0, 1, \leq, \ominus)$ be a regular DP. For any $a, b \in K$ with $b \leq 1\ominus a$, put $a\oplus b = 1\ominus[(1\ominus a)\ominus b]$. Then $(K, 0, 1, \oplus)$ becomes an OA. A corollary: The categories $\mathscr{OA}$ and $\mathscr{DP}_{reg}$ are isomorphic.*

Proof. Prior to verifying the axioms for an orthoalgebra observe that $b\ominus(b\ominus a) = a$ for any $a, b \in K$ with $a \leq b$. Indeed, $b\ominus a \leq b$ (from the definition of $\ominus$) and, by (dp2), $b\ominus(b\ominus a) = (b\ominus 0)\ominus(b\ominus a) = a\ominus 0 = a$.

Ad (oa1): Suppose that $a\oplus b$ is defined. This occurs exactly when $b \leq 1\ominus a$. Then $1\ominus b \geq 1\ominus(1\ominus a) = a$. Observe now that $(1\ominus a)\ominus b = (1\ominus b)\ominus a$. Indeed, making use of the axioms of a DP we obtain $(1\ominus a)\ominus b = (1\ominus a)\ominus[1\ominus(1\ominus b)] = (1\ominus b)\ominus a$. Applying the latter equality, we infer that $a\oplus b = 1\ominus[(1\ominus a)\ominus b] = 1\ominus[(1\ominus b)\ominus a] = b\oplus a$.

Ad (oa2): Suppose that $a\oplus(b\oplus c)$ is defined. Using (oa1), we obtain

$$a\oplus(b\oplus c) = (b\oplus c)\oplus a = 1\ominus[(1\ominus(b\oplus c))\ominus a] =$$

$$1\ominus\{[1\ominus[1\ominus((1\ominus b)\ominus c)]]\ominus a\} = 1\ominus[((1\ominus b)\ominus c)\ominus a] =$$

$$1\ominus[((1\ominus b)\ominus a)\ominus c] = (b\oplus a)\oplus c = (a\oplus b)\oplus c.$$

Ad (oa3): Suppose that $a \in K$. If we put $b = 1 \ominus a$, we obtain $a \oplus b = 1 \ominus [(1 \ominus a) \ominus b] = 1 \ominus (b \ominus b) = 1 \ominus 0 = 1$. We see that for $b = 1 \ominus a$ we have $a \oplus b = 1$. Suppose on the contrary that $a \oplus b = 1$ for an element $b \in K$. This means that $1 \ominus [(1 \ominus a) \ominus b] = 1$. It follows that $1 \ominus [1 \ominus ((1 \ominus a) \ominus b)] = 1 \ominus 1 = (1 \ominus 0) \ominus (1 \ominus 0) = 0 \ominus 0 = 0$. We infer that $(1 \ominus a) \ominus b = 0$ and therefore $1 \ominus a = (1 \ominus a) \ominus 0 = (1 \ominus a) \ominus ((1 \ominus a) \ominus b) = b$.

Ad (oa4): Suppose that $a \oplus a$ is defined. It follows that $a \leq 1 \ominus a$. Since K is supposed regular, we infer that $a = 0$. The proof is complete. □

If K is DP, we can also view it as a poset with a complementation if we endow it with a unary operation $'$ defined by putting $a' = 1 \ominus a$ $(a \in K)$. Then Th. 2.11 above implies: $(K, 0, 1, \oplus)$ is an orthoalgebra (with $a \oplus b = 1 \ominus [(1 \ominus a) \ominus b]$) if and only if $(K, 0, 1, \leq, ')$ is an orthoposet. (Recall that a bounded poset $(K, 0, 1, \leq, ')$ is called an orthoposet if $'$ is an involutive antiisomorphism such that $a \vee a' = 1$ for all $a \in K$.)

In our next result we find conditions for a DP to become an orthomodular poset, an orthomodular lattice, or a Boolean algebra. This result may be useful in the logico-algebraic foundation of quantum theories, or elsewhere. (The proofs are easily available from Th. 2.11 above, Th. 2.11 of [8] and the rudiments on orthomodular posets (see e.g. [12])).

Theorem 2.12 : *Let K be a regular DP and let $'$ denote a unary operation on K such that, for any $a \in K$, $a' = 1 \ominus a$.*

1. *K is an orthomodular poset if and only if the following condition* (OP) *is fulfilled: If $b \leq 1 \ominus a$, then the supremum $a \vee b$ exists in K and equals $1 \ominus [(1 \ominus a) \ominus b]$. A corollary: The full subcategory of $\mathscr{DP}_{reg}$ consisting of exactly those objects that fulfil the condition* (OP) *is isomorphic to the category of orthomodular posets.*

2. *K is an orthomodular lattice if and only if it is a lattice. A corollary: The full subcategory of $\mathscr{DP}_{reg}$ consisting of exactly those objects that are lattices is isomorphic to the category of orthomodular lattices with the OA morphisms.*

3. *K is a Boolean algebra if and only if the following condition* (B) *is fulfilled: For any pair $a, b \in K$ there is a triple $e, f, g \in K$ such that $e \leq 1 \ominus f$, $f \leq 1 \ominus g$, $g \leq 1 \ominus e$ and $f = a \ominus e$, $g = b \ominus e$. A corollary: The full subcategory of $\mathscr{DP}_{reg}$ whose objects fulfil* (B) *is isomorphic to the category of Boolean algebras.*

Let us note in conclusion of this note that the isomorphism of regular DPs and OAs allows us to unify the investigation of states and observables in OAs. Indeed, if K is an OA and if K is viewed as a DP, then a state on K may be identified with a DP morphism of K into V_f (see Ex. 2.2) and an observable on K may be identified with a DP morphism from $\mathscr{B}(R)$ into K, where $\mathscr{B}(R)$ is the Boolean algebra of Borel subsets of R (see [11]). Thus, both a state and an observable can be associated with the same mathematical entity. This might help in their description and investigation.

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