# On Fitting Operations

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**Abstract** The structure of fitting operations with respect to a given triangular norm is investigated. A special attention is paid to the case of basic t-norms. The connection between fitting property and Lipschitz property is stressed. Some examples are given.

**Key words** biresiduation, fitting operator, implication, residuation, triangular norm

## 1. Introduction

Several types of of many-valued logics on the unit interval [0,1] can be derived by means of triangular norms. Given a left-continuous t-norm **T**, the basic notion of a residuated lattice can be built up [3,7], including the residuation and biresiduation operators. Here the t-norm **T** plays the role of a conjunction, the corresponding residual operator $\mathbf{I}_T$ corresponds to an implication and the biresidual operator $\mathbf{E}_T$ models the equivalence. For practical purposes, often several other operations on the residuated lattice should be introduce. It is natural to require that they fit with the underlaying biresiduation (equivalence) - such operations are called fitting (T-fitting) operations [7]. The aim of this paper is to investigate the structure of fitting operations especially with respect to the basic triangular norms.

## 2. Preliminaries and basic properties

Let **T** be a left-continuous triangular norm [5,8], i.e. **T** is a left-continuous $[0,1]^2 \rightarrow [0,1]$ commutative, associative, non-decreasing mapping such that $T(x,1) = x$ for all $x \in [0,1]$. The associativity of **T** allows to extend it to be an n-ary operation, $n \geq 3$, too, namely

$$T(x_1,\ldots,x_n) = T(T(x_1,\ldots,x_{n-1}),x_n) .$$

If $x_1 = \ldots = x_n = x$, we put $T(x_1,\ldots,x_n) = x^{(n)}$ (if no confusion with respect to the t-norm we are dealing with can occur).

We recall some basic triangular norms:

- minimum $T_M(x,y) = \min(x,y)$ ;
- product $T_P(x,y) = xy$ ;
- Lukasiewicz t-norm $T_L(x,y) = \max(0,x+y-1)$ ;
- the weakest t-norm $T_W(x,y) = 0$ whenever $\max(x,y) < 1$ .

Note that up to the weakest t-norm $T_W$ all introduced t-norms are continuous and hence left-continuous. An example of a non-continuous left-continuous t-norm is the Fodor t-norm $T_F$,

$$T_F(x,y) = \begin{cases} \min(x,y) & \text{if } x+y > 1 \\ 0 & \text{otherwise} \end{cases} .$$

An important role in the t-norm theory play the triangular norms with additive generators. As far as the left-continuity of a t-norm T possessing an additive generator $f$ implies immediately its continuity [4], we will deal with continuous additive generators only.

Let $f: [0,1] \to [0,\infty]$ be a continuous strictly decreasing mapping such that $f(1) = 0$. Then $f$ generates a t-norm T via

$$T(x,y) = f^{-1}(\min(f(0), f(x)+f(y))) .$$

Note that the product t-norm $T_P$ is generated by an additive generator $f_P(x) = -\log x$ , while the Lukasiewicz t-norm $T_L$ is generated by an additive generator $f_L(x) = 1-x$ . Each t-norm generated by an unbounded continuous additive generator $f$ is isomorphic with the product t-norm and it is called a strict t-norm. Similarly, each t-norm generated by a bounded continuous additive generator is isomorphic with the Lukasiewicz t-norm and it is called a nilpotent t-norm. Additive continuous generators are determined uniquely up to a positive multiplicative constant, not influencing any property of the corresponding t-norm.

For a given t-norm T, the residual implicator $I_T$ is a $[0,1]^2 \to [0,1]$ mapping defined via [1,2,3,7]

$$I_T(x,y) = \sup(z;\ T(x,z) \le y) .$$

Note that the property $I_T(x,y) = 1$ if and only if $x \le y$ is fulfilled for left-continuous t-norms (in fact, the border continuity of T is a necessary and sufficient condition)and therefore we will deal with left-continuous t-norms in what follows only. For the basic t-norms we have the following residual implicators:

- for $\mathbf{T}_M$: $\mathbf{I}_M(x,y) = \begin{cases} 1 & \text{if } x \le y \\ y & \text{otherwise} \end{cases}$ (Goedel implication) ;
- for $\mathbf{T}_P$: $\mathbf{I}_P(x,y) = \begin{cases} 1 & \text{if } x \le y \\ y/x & \text{otherwise} \end{cases}$ (Goguen implication) ;
- for $\mathbf{T}_L$: $\mathbf{I}_L(x,y) = \min(1, 1-x+y)$ (Lukasiewicz implication) .

For a t-norm $\mathbf{T}$ generated by a continuous additive generator $f$, the corresponding residual implicator is defined via

$$\mathbf{I}_T(x,y) = f^{-1}(\max(0, f(y)-f(x))) \ .$$

For more details about residual implicators we recommend [1].

For a given t-norm $\mathbf{T}$, the biresidual operator $\mathbf{E}_T$ is a $[0,1]^2 \to [0,1]$ mapping defined via

$$\mathbf{E}_T(x,y) = \min(\mathbf{I}_T(x,y), \mathbf{I}_T(y,x)) \ .$$

Note that the minimum in the above definition can be replaced by an arbitrary t-norm with no influence (as far as at least one of arguments is equal to 1). Further, we have also

$$\mathbf{E}_T(x,y) = \mathbf{I}_T(\max(x,y), \min(x,y)) \ .$$

We recall basic biresidual operators corresponding to the above introduced basic t-norms:

- $\mathbf{E}_M(x,y) = \begin{cases} 1 & \text{if } x=y \\ \min(x,y) & \text{otherwise} \end{cases}$ ;
- $\mathbf{E}_P(x,y) = \min(x,y)/\max(x,y) = \exp(-|\log x - \log y|)$ ,

where $0/0 = 1$, resp. $\infty - \infty = 0$ ;

- $\mathbf{E}_L(x,y) = 1 - |x-y|$ .

If $\mathbf{T}$ is generated by a continuous additive generator $f$, then

$$\mathbf{E}_T(x,y) = f^{-1}(|f(x)-f(y)|) \ .$$

Now, we are able to introduce the notion of a fitting operation, see [7].

**Definition 1** *Let* $\mathbf{K}: [0,1]^n \to [0,1]$ *be some n-ary operation for* $n \in \mathbf{N}$. *Let* $\mathbf{T}$ *be a left-continuous t-norm. We say that* $\mathbf{K}$ *is a* $\mathbf{T}$*-fitting operation if there exist integers* $k_1, \dots, k_n \in \mathbf{N}$ *such that for arbitrary elements* $a_1, \dots, a_n, b_1, \dots, b_n \in [0,1]$ *we have*

$T(E_T(a_1,b_1)^{(k_1)},\ldots,E_T(a_n,b_n)^{(k_n)}) \le E_T(K(a_1,\ldots,a_n),K(b_1,\ldots,b_n))$ (1).

### 3. Basic properties of fitting operations

In this section, we give some general properties of fitting operations. The monotonicity of T ensures the validity of the next proposition.

**Proposition 1** *K is a T-fitting operation if and only if there is* $k \in \mathbf{N}$ *such that inequality* (1) *holds with* $k_1=\ldots=k_n=k$ .

For n-ary operations defined by means of some associative binary operation **K** we have the following result.

**Theorem 1** *Let* **K** *be an associative binary operation on the unit interval and let* $K_n$, n=2,3,..., *be the corresponding n-ary operation (i.e.* $K=K_2$*). Let* **T** *be a given left-continuous t-norm. Then* **K** *is a* T-*fitting operation if and only if all* $K_n$, n=2,3,..., *are* T-*fitting operations.*

*Proof.* It is enough to show that if **K** is T-fitting then also $K_3$ is T-fitting (and then the rest of the proof follows by induction). Suppose that **K** is T-fitting and let $k \in \mathbf{N}$ be the corresponding constant from Proposition 1. Then

$$E_T(K_3(a_1,a_2,a_3),K_3(b_1,b_2,b_3)) = E_T(K(K(a_1,a_2),a_3),K(K(b_1,b_2),b_3)) \ge$$
$$T(E_T(K(a_1,a_2),K(b_1,b_2))^{(k)},E_T(a_3,b_3)^{(k)}) \ge$$
$$T(T(E_T(a_1,b_1)^{(k)},E_T(a_2,b_2)^{(k)})^{(k)},E_T(a_3,b_3)^{(k)}) =$$
$$T(E_T(a_1,b_1)^{(k^2)},E_T(a_2,b_2)^{(k^2)},E_T(a_3,b_3)^{(k)}) ,$$

and consequently $K_3$ is a T-fitting operation (with constants $k_1=k_2=k^2$ and $k_3=k$) . □

Now, we are able to show that for each left-continuous t-norm **T**, both **T** and $T_M$ are T-fitting operations (otherwise the concept of fitting operations wouldn't be sound).

**Theorem 2** *Let* **T** *be a left-continuous t-norm. Then* **T**, $T_M$ *and all their n-ary extensions,* n=2,3,..., *are* T-*fitting operations.*

*Proof.* Due to Theorem 1, it is enough to show that **T** as a binary operation on the unit interval is a T-fitting operation with k=1 (see Proposition 1).
Let $a_1 \le b_1$ and $a_2 \le b_2$. Put $u=\mathbf{E}_T(a_1, b_1)$ and $v= \mathbf{E}_T(a_2, b_2)$. Then

$\mathbf{T}(b_1, u)=a_1$ and $\mathbf{T}(b_2, v)=a_2$ and consequently due to the associativity of **T**

we have $\mathbf{T}(a_1, a_2)=\mathbf{T}(\mathbf{T}(b_1, b_2), \mathbf{T}(u, v))$. It follows that

$\mathbf{T}(u, v) \le \mathbf{E}_T(\mathbf{T}(a_1, a_2), \mathbf{T}(b_1, b_2))$ proving the inequality (1) for this case.

Similar is the case $a_1 \ge b_1$ and $a_2 \le b_2$.
Now, let $a_1 \le b_1$ and $a_2 \ge b_2$ and let u and v be defined as above. Then

$\mathbf{T}(b_1, u)=a_1$ and $\mathbf{T}(a_2, v)=b_2$. Suppose that $\mathbf{T}(a_1, a_2) \le \mathbf{T}(b_1, b_2)$ (the opposite inequality can be treated similarly). Then using the same arguments as in the first case, we have $\mathbf{T}(u, v) \le \mathbf{E}_T(\mathbf{T}(a_1, b_2), \mathbf{T}(b_1, a_2))=z$. Now,

$\mathbf{T}(\mathbf{T}(b_1, b_2), z) \le \mathbf{T}(\mathbf{T}(b_1, a_2), z)=\mathbf{T}(a_1, b_2) \le \mathbf{T}(a_1, a_2)$ ensures the result.

Indeed, $\mathbf{T}(u, v) \le z \le \mathbf{E}_T(\mathbf{T}(a_1, a_2), \mathbf{T}(b_1, b_2))$ proves inequality (1).
Using similar arguments, the rest of the proof for **T** and the statement for $\mathbf{T}_M$ can be shown. □

For a given left-continuous t-norm **T**, the mapping $\mathbf{n}_T: [0,1] \longrightarrow [0,1]$ defined by $\mathbf{n}_T(x) = \mathbf{I}_T(x, 0)$ is a T-negation. it is easy to see that $\mathbf{n}_T$ is decreasing and that $\mathbf{n}_T(1) = 0$, $\mathbf{n}_T(0) = 1$.

**Theorem 3** *Let* **T** *be a left-continuous t-norm. Then the corresponding T-negation* $\mathbf{n}_T$ *is a T-fitting operation.*

*Proof.* Without any loss of generality, we can assume a<b and then $\mathbf{n}_T$ is T-fitting operation whenever $\mathbf{I}_T(b, a) \le \mathbf{I}_T(\mathbf{I}_T(a, 0), \mathbf{I}_T(b, 0))$. Put $z=\mathbf{I}_T(b, a)$, $u=\mathbf{I}_T(a, 0)$ and $v=\mathbf{I}_T(b, 0)$. Then $\mathbf{T}(b, z)=a$, $\mathbf{T}(a, u)=0$ and consequently $0=\mathbf{T}(a, u)=\mathbf{T}(\mathbf{T}(b, z), u)=\mathbf{T}(b, \mathbf{T}(z, u))$ and thus $\mathbf{T}(z, u) \le v$. But then $z \le \mathbf{I}_T(u, v)$, q.e.d. □

## 4. Unary fitting operations

In this section, we will investigate unary fitting operation. Especially important is the problem of fitting negations, i.e. decreasing mappings on the unit interval mapping 1 into 0 and vice versa.

i) $T_M$-fitting unary operations:

$K:[0,1]\to[0,1]$ is $T_M$-fitting only if for all a, b $\in$ [0,1] it is $E_M(a,b)\le E_M(K(a),K(b))$. If a=b then the last inequality is obviously fulfilled. Without any loss of generality, we may suppose that a<b, i.e. $a\le E_M(K(a),K(b))$. This means that either $K(a)=K(b)$ or $\min(K(a),K(b))\ge a$. Then if $K(a)<a$, we have $K(b)=K(a)$ for all $b\ge a$. For $x\in\,]K(a),a[$ then $K(a)<x\le E_M(K(x),K(a))$ requires $K(x)=K(a)$. Consequently, K is $T_M$-fitting if and only if $K(x)\ge x$ for all $x\le u=\inf(z;K(z)<z)$ and $K(x)=u$ for all $x>u$. This means that the only $T_M$-fitting negation is just the $T_M$-negation $n_M$, $n_M(x)=0$ for all $x>0$.

ii) $T_L$-fitting unary operations:

K is $T_L$-fitting if and only if there is $k\in\mathbf{N}$ such that for all a, b $\in$ [0,1] it is $(1-|a-b|)^{(k)}\le 1-|K(a)-K(b)|$. The last inequality is equivalent with the Lipschitz property

$$|K(a)-K(b)|\le k|a-b| .$$

Recall that a real function K fulfills the Lipschitz property if there is some positive constant c such that for all x, y from the domain of K it is $|K(x)-K(y)|\le c|x-y|$.

Hence K is $T_L$-fitting if and only if K fulfills the Lipschitz property. If K is differentiable on ]0,1[, then K is $T_L$-fitting if and only if its derivative K' is bounded on ]0,1[. Further, the continuity of K is a necessary condition for K to be $T_L$-fitting (see also [7]). Recall that the $T_L$-negation $n_L$ is the usual [0,1]-valued logic negation, $n_L(x)=1-x$, and that $n_L$ is obviously $T_L$-fitting. Note that the negation $K(x)=1-x^p$ is $T_L$-fitting for all p>0, while the strong negation $K(x)=(1-x^p)^{1/p}$ is $T_L$-fitting only when p=1 (and then $K=n_L$).

iii) $T_P$-fitting unary operations:

K is $T_P$-fitting if and only if there is $k\in\mathbf{N}$ such that for all a, b $\in$ [0,1] it is $(\exp(-|\log a-\log b|))^k\le\exp(-|\log K(a)-\log K(b)|)$, i.e. $|\log K(a)-\log K(b)|\le k|\log a-\log b|$. Put $u=\log a$ and $v=\log b$. Then the last inequality can be rewritten into

$$|\log K(\exp u)-\log K(\exp v)|\le k|u-v| ,$$

i.e. the composite function $\log\circ K\circ\exp$ is Lipschitz on $[-\infty,0]$. Hence a unary operation K is $T_P$ fitting if and only if the composite function $\log\circ K\circ\exp$ fulfills the Lipschitz property on $[-\infty,0]$. Again as in the previous case, the continuity of $K$ (up to the point 0) is a necessary condition for the $T_P$-fitness. $K$ is $T_P$-fitting e.g. if $\log\circ K\circ\exp$ is differentiable on $]-\infty,0[$ and its derivative is bounded. So, e.g., $K(x) = x^p$ is $T_P$- fitting for each $p>0$. Note that the only $T_P$-fitting negation is just the $T_P$-negation $n_P=n_M$.

iv) the case of t-norms with additive generators:
Let $f$ be a continuous additive generator of a t-norm $T$. Then the unary operation $K$ is T-fitting if and only if there is some $k \in \mathbf{N}$ such that for all a, b $\in [0,1]$ it is $(f^{-1}(|f(a)-f(b)|))^{(k)}\le f^{-1}(|f(K(a))-f(K(b))|)$, i.e.

$$|f(K(a))-f(K(b))| \le k|f(a)-f(b)| .$$

Similarly as in the case of the product t-norm $T_P$ the last inequality means the Lipschitz property of the composite function $f\circ K\circ f^{-1}$ on the range of the generator $f$. Hence only continuous unary operations $K$ (possibly up to the point 0 in the case of strict t-norms) are appropriate candidates for T-fitting operations. Note that in the case of strict t-norm $T$, the only T-fitting negation is the corresponding T-negation $n_T = n_M$ .

### 5. Fitting binary operations

The only $T_M$-fitting triangular norm is $T_M$ itself, while each t-conorm is $T_M$-fitting operation (note that t-conorms are commutative associative non-decreasing binary operations on [0,1] with 0 as neutral element, see [5,8].
Let $K:[0,1]^2\longrightarrow[0,1]$ be some binary operation. It is $T_L$-fitting if and only if there is some $k \in \mathbf{N}$ such that for all $a_1,a_2,b_1,b_2 \in [0,1]$ we have $T_L(1-|a_1-b_1|,1-|a_2-b_2|)^{(k)}\le 1 - |K(a_1,a_2)-K(b_1,b_2)|$ , i.e.

$$|K(a_1,a_2)-K(b_1,b_2)| \le k(|a_1-b_1| + |a_2-b_2|) .$$

Hence a binary operation $K$ is $T_L$-fitting if and only if $K$ fulfills the Lipschitz property (for two-place functions). Then the continuity of $K$ is a necessary condition for $T_L$-fitness (see also [6]) and if $K$ is differentiable, then the boundedness of the first partial derivatives of $K$ (on the open unit square) ensures that $K$ is $T_L$-fitting. Taking into account the t-norms and t-conorms with continuous additive generators, then $T$ is $T_L$-fitting if and only if the inverse $f^{-1}$ of the corresponding additive generator $f$ fulfills the Lipschitz property. So, e.g., if the first derivative of $f^{-1}$ is bounded on $]0,f(0)[$, then the corresponding **T** is $\mathbf{T}_L$-fitting. This is e.g. the case of the product t-norm $\mathbf{T}_P$, where $f^{-1}(x) = \exp(-x)$ for $x \in [0,\infty]$ and $|df^{-1}(x)/dx| = |-\exp(-x)| \le 1$. A similar claim can be applied to the t-conorms generated by a continuous additive generator $g: [0,1] \rightarrow [0,\infty]$ (continuous strictly increasing mapping with $g(0) = 0$). The corresponding t-conorm **S** is $\mathbf{T}_L$-fitting if and only if the function $g^{-1}$ fulfills the Lipschitz property on $]0,g(1)[$.

Similar is the situation with $\mathbf{T}_P$-fitting binary operations and more generally with T-fitting binary operations, where **T** is generated by a continuous additive generator $f$.

**Theorem 4** *Let* **T** *be a t-norm generated by a continuous additive generator $f$. Then a binary operation* $\mathbf{K}: [0,1]^2 \rightarrow [0,1]$ *is* **T**-*fitting if and only if the binary operation* $\mathbf{H}: [0,f(0)]^2 \rightarrow [0,f(0)]$ *defined via*

$$\mathbf{H}(x,y) = f(\mathbf{K}(f^{-1}(x),f^{-1}(y)))$$

*fulfills the Lipschitz property on* $[0,f(0)]$. *The necessary condition for the* **T**-*fitness is the continuity of* **K** *(possibly up to the case when* $0 \in \{x,y\}$ *if* **T** *is a strict t-norm) while the sufficient condition is the boundedness of the first partial derivatives of* **H** *on* $]0,f(0)[$ . □

Note that if **K** is a t-norm (or t-conorm) generated by a continuous additive generator $h$, then **K** is **T**-fitting if and only if the composite function $f \circ h^{-1}$ fulfills the Lipschitz property (here $f$ is a continuous additive generator of **T**).

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