TRIANGULAR NORM–BASED ADDITIONS OF FUZZY NUMBERS AND PRESERVING OF SIMILARITY

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ABSTRACT.

In the paper the notions of similar fuzzy numbers and preserving of similarity are introduced. Using of these notions enables to investigate preserving of the type of fuzzy numbers with respect to the t-norm-based arithmetical operations, especially in the case of fuzzy numbers with unbounded supports. The conditions for preserving of similarity of fuzzy numbers with unbounded supports with respect to the additions based on continuous Archimedean t-norms are given.

Key words : fuzzy number, addition of fuzzy numbers, triangular norm

1. INTRODUCTION

In the beginning we recall the basic notions which will be used throughout the paper.

A triangular norm (t-norm) T is a bivariate function $T\colon [0,1]^2 \to [0,1]$ which is associative, commutative, non-decreasing, and $T(x,1) = x$ for each $x \in [0,1]$.

A fuzzy quantity is any fuzzy subset in the universe of real numbers $\mathbb{R}$. A fuzzy quantity A is represented by its membership function $\mu_A : \mathbb{R} \to [0,1]$.

For a given t-norm T, the membership function $\mu_{A \underset{T}{\oplus} B}$ of a t-norm-based sum of fuzzy quantities A, B is defined as follows:

$$\mu_{A \underset{T}{\oplus} B}(z) = \sup_{z=x+y} T\left(\mu_A(x), \mu_B(y)\right), \quad z \in \mathbb{R}, \tag{1}$$

or in the modified form by

$$\mu_{A \underset{T}{\oplus} B}(z) = \sup_{x \in \mathbb{R}} T\left(\mu_A(x), \mu_B(z-x)\right), \quad z \in \mathbb{R}. \tag{2}$$

If T is a continuous Archimedean t-norm, i.e., a continuous t-norm with $T(x,x) < x$ for each $x \in (0,1)$, then there exists a continuous, strictly decreasing function $f\colon [0,1] \to [0,\infty]$, $f(1) = 0$, such that

$$T(x,y) = f^{-1}\left(\min\left(f(0), f(x) + f(y)\right)\right), \quad x, y \in [0,1],$$

where f^{-1} is an inverse function of f.

A function f is called an additive generator of a t-norm T, and it is determined uniquely up to a positive multiplicative constant. In the case of a continuous Archimedean t-norm T, the membership function of the sum $A \underset{T}{\oplus} B$ is given by

$$(3) \qquad \mu_{A \underset{T}{\oplus} B}(z) = f^{-1}\left(\min\left(f(0), \inf_{x \in \mathbb{R}} (f \circ \mu_A(x) + f \circ \mu_B(z-x))\right)\right), \quad z \in \mathbb{R}.$$

At the following part of the paper, we will restrict ourselves to fuzzy numbers.

A fuzzy quantity A is said to be a fuzzy number if the membership function μ_A is continuous and for all $\alpha \in (0,1]$ the corresponding α-cuts $A_\alpha = \{x \in \mathbb{R}; \mu_A(x) \geq \alpha\}$ are convex compact sets , and $A_1 = \{a\}, a \in \mathbb{R}$.

The point $a \in \mathbb{R}$ is called the peak of a fuzzy number A. The set of all considered fuzzy numbers (regardless of the peak) will be denoted by $\mathcal{A}$, and the set of all fuzzy numbers with the peak in the point a by $\mathcal{A}^a$.

DEFINITION 1. *A fuzzy number $B \in \mathcal{A}^b$ is similar to a fuzzy number $A \in \mathcal{A}^a$ if there exists a mapping φ,*

$$\varphi(x) = \begin{cases} \delta(x-b)+a, & \text{for } x \leq b, \\ \epsilon(x-b)+a, & \text{for } x \geq b, \end{cases}$$

where $\delta, \epsilon > 0$, such that

$$\mu_B(x) = \mu_A \circ \varphi(x), \quad x \in \mathbb{R}.$$

The fact that a fuzzy number B is similar to a fuzzy number A will be denoted by $B \overset{\delta,\epsilon}{\sim} A$, or briefly by $B \sim A$. The positive number $\delta(\epsilon)$ will be called a coefficient of similarity of the left (right) parts of the membership functions μ_B and μ_A.

The following properties can be easy proved.

PROPOSITION 1. *The relation of similarity $\sim$ is an equivalence on the set $\mathcal{A}$.*

The class of all fuzzy numbers which are similar to a fuzzy number A will be denoted by $[A]$.

A crisp real number a can be regarded as a fuzzy quantity with a membership function

$$\mu_{\{a\}}(x) = \begin{cases} 1, & \text{for } x = a, \\ 0, & \text{for } x \neq a. \end{cases}$$

Using this fact, we can formulate the following statements.

PROPOSITION 2. *Let T be a t-norm.*

(i) If $A \in \mathcal{A}^a$, $b \in \mathbb{R}$ and $B = A \underset{T}{\oplus} b$, then $B \in \mathcal{A}^{a+b}$ and $B \overset{1,1}{\sim} A$.

(ii) For each $A \in \mathcal{A}^a$ there exists $A^0 \in \mathcal{A}^0$ such that $A = A^0 \underset{T}{\oplus} a$, and $A^0 \overset{1,1}{\sim} A$. The fuzzy number A^0 is determined uniquely.

COROLLARY 1. *If $A \in \mathcal{A}^a$ and $B \in \mathcal{A}^b$, then*

$$A \underset{T}{\oplus} B \in \mathcal{A}^{a+b}, \quad \text{and} \quad A \underset{T}{\oplus} B = (A^0 \underset{T}{\oplus} B^0) \underset{T}{\oplus} (a+b),$$

where $A^0, B^0 \in \mathcal{A}^0$ and $A^0 \underset{T}{\oplus} B^0 \overset{1,1}{\sim} A \underset{T}{\oplus} B$.
The fuzzy numbers A^0, B^0 are determined uniquely.

Due to these properties we can examine only fuzzy numbers with the peak in the point zero because it is often simpler.
Note that similarity of fuzzy numbers $A, B \in \mathcal{A}^0$ means that

$$\mu_B(x) = \begin{cases} \mu_A(\delta x), & \text{for } x \le 0, \\ \mu_A(\epsilon x), & \text{for } x \ge 0, \end{cases}$$

for some $\delta, \epsilon > 0$.

2. Preserving of similarity

If we work with fuzzy numbers of the certain type, it is reasonable to require their sum to be of the same type, too. This property can be described by means of similarity.
We can say that the fuzzy numbers A, B are of the same type if they are similar. Our aim is to find the conditions under which for each $A, B \in \mathcal{A}$, $B \sim A$, the T-sum $A \underset{T}{\oplus} B$ will be similar to A (and, of course, to B).

DEFINITION 2.

(1) *Let $A \in \mathcal{A}$. The addition $\underset{T}{\oplus}$ preserves A-similarity if for each $B \in [A]$ the t-norm-based sum $A \underset{T}{\oplus} B$ also belongs to $[A]$.*

(2) *The addition $\underset{T}{\oplus}$ preserves similarity if it preserves A-similarity for each $A \in \mathcal{A}$.*

At first, consider the weakest t-norm T_W which is given by

$$T_W(x,y) = \begin{cases} \min(x,y), & \text{if } \max(x,y) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

It is known [8] that in the case of the t-norm T_W, for each $A, B \in \mathcal{A}$, it holds:

$$\mu_{A \underset{T_W}{\oplus} B}(z) = \max(\mu_A(z), \mu_B(z)), \quad z \in \mathbb{R}.$$

Let $A \in \mathcal{A}^0$. Then for each $B \in \mathcal{A}^0, B \overset{\delta,\epsilon}{\sim} A$, we get

$$\mu_{A \underset{T_W}{\oplus} B}(z) = \begin{cases} \mu_A(\delta^* z), & \text{for } z \le 0 \\ \mu_A(\epsilon^* z), & \text{for } z \ge 0, \end{cases}$$

where $\delta^* = \min(1, \delta)$ and $\epsilon^* = \min(1, \epsilon)$, which proves that the addition $\underset{T_W}{\oplus}$ preserves A-similarity. Since this property holds for each $A \in \mathcal{A}^0$, the addition $\underset{T_W}{\oplus}$ preserves similarity. And this is precisely the assertion of the following proposition.

PROPOSITION 3. *The addition based on the t-norm T_W preserves similarity. Moreover, if $A, B \in \mathcal{A}$ and $B \overset{\delta,\epsilon}{\sim} A$, then*

$$A \underset{T_W}{\oplus} B \overset{\delta^*,\epsilon^*}{\sim} A,$$

where $\delta^* = \min(1, \delta)$ and $\epsilon^* = \min(1, \epsilon)$.

From now, let T be a continuous t-norm.

An important continuous t-norm is the t-norm T_M, given by

$$T_M(x, y) = \min(x, y), \quad x, y \in [0, 1].$$

T_M is not an Archimedean t-norm, it has no additive generator. Therefore we will examine it separately.

PROPOSITION 4. *The addition of fuzzy numbers based on the t-norm T_M preserves similarity.*

Proof. By Proposition 2, it is enough to prove that for each $A, B \in \mathcal{A}^0$, $B \sim A$, the sum $C = A \underset{T_M}{\oplus} B \sim A$.

Let $A, B \in \mathcal{A}^0$, $B \overset{\delta,\epsilon}{\sim} A$, and let $A_\alpha = [a_\alpha^A, b_\alpha^A]$, $\alpha > 0$ be α-cuts of the fuzzy number A. Then α-cuts of B have the form

$$B_\alpha = \left[\frac{a_\alpha^A}{\delta}, \frac{b_\alpha^A}{\epsilon}\right].$$

For α-cuts C_α of the sum $C = A \underset{T_M}{\oplus} B$ it holds $C_\alpha = A_\alpha + B_\alpha$, $\alpha > 0$. It means that

$$C_\alpha = \left[\left(1 + \frac{1}{\delta}\right) a_\alpha^A, \left(1 + \frac{1}{\epsilon}\right) b_\alpha^A\right], \quad \alpha > 0.$$

Therefore $C \overset{\delta^*,\epsilon^*}{\sim} A$, where

$$\delta^* = \frac{1}{1 + \frac{1}{\delta}}, \quad \epsilon^* = \frac{1}{1 + \frac{1}{\epsilon}},$$

and the proof is over.

□

We have shown not only that $C = A \underset{T_M}{\oplus} B \sim A$, but we have also found the coefficients of similarity, and we can quickly find the result of the addition. Moreover, let us note that δ^* is a harmonic average of 1 and δ, i.e., $\delta^* = \overline{(1, \delta)}_h$. The same holds for ϵ^*. This can be useful in the case of a greater number of summands.

We emphasize the obtained result in the following assertion.

COROLLARY 2. *If $A \in \mathcal{A}^a$, $B \in \mathcal{A}^b$ and $B \overset{\delta,\epsilon}{\sim} A$, then*

$$A \underset{T_M}{\oplus} B \in \mathcal{A}^{a+b}, \quad \textit{and} \quad A \underset{T_M}{\oplus} B \overset{\delta^*,\epsilon^*}{\sim} A, \quad \textit{where} \quad \delta^* = \frac{\delta}{\delta+1}, \quad \epsilon^* = \frac{\epsilon}{\epsilon+1}.$$

To illustrate the previous two assertions, consider the following numerical example.

EXAMPLE 1. Let A, B be fuzzy numbers with the membership functions :

$$\mu_A(x) = \begin{cases} x-1, & \text{for } x \in [1,2], \\ \frac{1}{2}(4-x), & \text{for } x \in [2,4], \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mu_B(x) = \begin{cases} \frac{1}{4}x, & \text{for } x \in [0,4], \\ 5-x, & \text{for } x \in [4,5], \\ 0, & \text{otherwise.} \end{cases}$$

It is only a matter of computation to show that $B \overset{\delta,\epsilon}{\sim} A$, where $\delta = \frac{1}{4}$ and $\epsilon = 2$. By Propositions 3 and 2, we get $C = A \underset{T_M}{\oplus} B \overset{\delta^*,\epsilon^*}{\sim} A$, where $\delta^* = \frac{1}{5}$ and $\epsilon^* = \frac{2}{3}$. Since $A \underset{T_M}{\oplus} B \in \mathcal{A}^6$, it holds

$$\mu_C(x) = \begin{cases} \mu_A(\frac{1}{5}(x-6)+2) = \frac{1}{5}(x-1), & \text{for } x \in [1,6], \\ \mu_A(\frac{2}{3}(x-6)+2) = \frac{1}{3}(9-x), & \text{for } x \in [6,9], \\ 0, & \text{otherwise.} \end{cases}$$

Now, let T be a continuous Archimedean t-norm.

From the assumption that all α-cuts $A_\alpha, \alpha > 0$ of a fuzzy number A are convex and compact sets, we get $\lim_{x \to +\infty} \mu_A(x) = 0$ and, the same for $x \to -\infty$.

There are two possibilities : either supp A is a bounded set, or supp A is an unbounded set.

Consider the first possibility : supp A is bounded.

In this case, a fuzzy number A is in fact so-called Left-Right or L–R fuzzy number. By [2], a fuzzy number A is called an L–R fuzzy number if the values $\mu_A(x)$ of the membership function can be calculated as follows :

$$\mu_A(x) = \begin{cases} L\left(\frac{a-x}{\alpha}\right), & \text{for } a-\alpha \le x \le a, \\ R\left(\frac{x-a}{\beta}\right), & \text{for } a \le x \le a+\beta, \\ 0, & \text{otherwise,} \end{cases}$$

where $a \in \mathbb{R}$, $\alpha, \beta > 0$, and $L, R \colon [0,1] \to [0,1]$ are the shape functions, which are continuous, non-decreasing, and $L(0) = R(0) = 1$, $L(1) = R(1) = 0$.

An L–R fuzzy number A is usually denoted by $A = (a, \alpha, \beta)_{LR}$. The real number a is said to be a peak of A and $\alpha(\beta)$ is the left (right) spread of A.

If A is an L–R fuzzy number and $B \in [A]$, then B is an L–R fuzzy number with the same shape functions L and R. Vice versa, every two L–R fuzzy numbers with the same shape functions L, R are similar. The formal proposition follows.

PROPOSITION 5.

(i) Let $A = (a, \alpha, \beta)_{LR}$, $B \in \mathcal{A}^b$ and $B \overset{\delta,\epsilon}{\sim} A$. Then $B = (b, \alpha^*, \beta^*)_{LR}$, where $\alpha^* = \frac{\alpha}{\delta}$, $\beta^* = \frac{\beta}{\epsilon}$.

(ii) If $A = (a, \alpha, \beta)_{LR}$, $B = (b, \alpha^*, \beta^*)_{LR}$, then $B \overset{\delta,\epsilon}{\sim} A$, where $\delta = \frac{\alpha}{\alpha^*}$, $\epsilon = \frac{\beta}{\beta^*}$.

The proof of this proposition is evident by applying the definitions.

From Proposition 5 follows that the addition $\underset{T}{\oplus}$ preserves similarity of $L{-}R$ fuzzy numbers iff it preserves the shape functions L, R.

The $L{-}R$-shape preserving additions were studied e.g. in [3], [4], [5], [7]. While the first three papers deal with preserving of linearity, in [7], due to a transformation principle, the sufficient conditions for preserving of any strictly decreasing shape functions L, R are given ([7], Theorem 4, Corollary 1).

The notion of similarity does not bring much new for $L{-}R$ fuzzy numbers, but it helps to overcome the problems in the case of fuzzy numbers with unbounded supports.

So, let us examine the second possibility mentioned above, i.e., the fuzzy numbers with unbounded supports. All conditions and results will only be formulated for the rigt parts of the membership functions. This can be done because the right and left parts of a membership function $\mu_{A \underset{T}{\oplus} B}$ are independent. The right part of the T-sum of fuzzy numbers A and B from $\mathcal{A}^0$, i.e., the values $\mu_{A \underset{T}{\oplus} B}(z)$ for $z \geq 0$, depend only on the right parts of the summands, i.e., on the values $\mu_A(x), \mu_B(y)$ for $x, y \geq 0$. The same holds for the left parts.

So, let $A \in \mathcal{A}^0$ and let supp A be unbounded from above, i.e., $\mu_A(x) > 0$ for each $x > 0$. Let $B \overset{\delta,\epsilon}{\sim} A$ and T be a continuous Archimedean t-norm.
By (3), using similarity B and A, the membership function μ_C of the T-sum $C = A \underset{T}{\oplus} B$ can be expressed as follows :

$$\mu_C(z) = f^{-1}\left(\min\left(f(0), \inf_{x \in [0,z]} (f \circ \mu_A(x) + f \circ \mu_A(\epsilon(z-x)))\right)\right), \quad z > 0,$$

or in the form :

$$f \circ \mu_C(z) = \min\left(f(0), \inf_{x \in [0,z]} (f \circ \mu_A(x) + f \circ \mu_A(\epsilon(z-x)))\right), \quad z > 0. \tag{5}$$

The addition $\underset{T}{\oplus}$ preserves A-similarity (of the right parts) iff for each $\epsilon > 0$ there exists $\tau > 0$ such that $\mu_C(z) = \mu_A(\tau z)$ for each $z > 0$. In that case, we get

$$f \circ \mu_A(\tau z) = \min\left(f(0), \inf_{x \in [0,z]} (f \circ \mu_A(x) + f \circ \mu_A(\epsilon(z-x)))\right), \quad z > 0. \tag{6}$$

Put $f \circ \mu_A|_{[0,\infty)} = g$. Then

$$g(\tau z) = \min\left(f(0), \inf_{x \in [0,z]} (g(x) + g(\epsilon(z-x)))\right), \quad z > 0. \tag{7}$$

Note that $g\colon [0,\infty) \to [0,\infty]$ is a continuous, non-decreasing function with $g(0) = 0$.

We have asked T to be a continuous Archimedean t-norm. Each continuous Archimedean t-norm T is either strict or nilpotent. The following two theorems will be formulated for strict t-norms.

THEOREM 1. *Let $A \in \mathcal{A}^0$ and $\mu_A(x) > 0$ for each $x \geq 0$. Let T be a strict t-norm with an additive generator f, and let the addition $\underset{T}{\oplus}$ preserves A-similarity.*

(i) If the function $f \circ \mu_A|_{[0,\infty)}$ is convex, then

$$\mu_A(z) = f^{-1}(az^s) \quad \textit{for each } z > 0 \textit{ and some } a > 0, s \geq 1.$$

(ii) If the function $f \circ \mu_A|_{[0,\infty)}$ is concave, then

$$\mu_{A \underset{T}{\oplus} A}(z) = \mu_A(z) \quad \textit{for each } z \geq 0.$$

Proof. (i) By the assumption, T is a strict t-norm, i.e., its additive generator is an unbounded function. Therefore (7) can be rewritten into the form :

$$g(\tau z) = \inf_{x\in[0,z]} \left(g(x) + g\left(\epsilon(z-x)\right)\right), \quad z > 0. \tag{8}$$

We have used notation $g = f \circ \mu_A|_{[0,\infty)}$.
Consider $B = A$, i.e, $\epsilon = 1$. Denote by τ^* the number τ corresponding in (8) to $\epsilon = 1$. We get

$$g(\tau^* z) = \inf_{[0,z]} \left(g(x) + g(z-x)\right), \quad z > 0. \tag{9}$$

Put $h(x) = g(x) + g(z-x)$, $x \in [0,z]$. The function h is continouos, convex, $h(0) = h(z) = g(z)$. Moreover, h is a symmetric function, i.e., $h(x) = h(z-x)$, $x \in [0,z]$. Therefore

$$\inf_{x\in[0,z]} h(x) = h\left(\frac{z}{2}\right) = 2g\left(\frac{z}{2}\right),$$

and from (9), we get

$$g(\tau^* z) = 2g\left(\frac{z}{2}\right), \quad z > 0. \tag{10}$$

Under given assumptions, the function g is strictly increasing and unbounded. The number $\tau^* > \frac{1}{2}$. Define a function u by

$$u\colon (0,\infty) \to (\frac{1}{2},\infty), \quad u(s) = 2^{\frac{1-s}{s}}.$$

The function u is a strictly decreasing, continuous bijection. Therefore for τ^* there exists a unique number $s \in (0,\infty)$ such that $\tau^* = 2^{\frac{1-s}{s}}$. If we substitute this value to (10), and denote $\lambda = 2^{\frac{1}{s}}$, $\frac{z}{2} = t$, we get

$$g(\lambda t) = \lambda^s g(t), \quad t > 0. \tag{11}$$

It is known [1] that for a given $s > 0$, the only non-negative, continuous solutions of the functional equation (11) in the interval $(0, \infty)$, are the functions g,

$$g(t) = at^s, \quad a > 0.$$

These functions are convex iff $s \geq 1$. Using $g = f \circ \mu_A|_{[0,\infty)}$ and $z = 2t$, we get our claim.

(ii) If the function $g = f \circ \mu_A|_{[0,\infty)}$ is concave, then the function h, introduced in (i), is also concave. Therefore

$$\inf_{x \in [0,z]} h(x) = h(0) = g(z),$$

and from (9), it follows

$$g(\tau^* z) = g(z), \quad z > 0. \tag{12}$$

The equality (12) is true iff $\tau^* = 1$, and in that case

$$\mu_{A \underset{T}{\oplus} A}(z) = \mu_A(z), \quad z > 0.$$

□

If the same assumptions as in Theorem 1, (ii) hold for the left part of the membership function μ_A, then $A \underset{T}{\oplus} A = A$, which means that A is an idempotent with respect to the addition $\underset{T}{\oplus}$.

THEOREM 2. *Let $A \in \mathcal{A}^0$, $\mu_A > 0$ for each $x \geq 0$. Let T be a strict t-norm with an additive generator f.*

(i) If for each $x > 0$

$$\mu_A(x) = f^{-1}(ax^s),$$

where a, s are any real numbers, $a > 0$, $s \geq 1$,
or

(ii) if $f \circ \mu_A|_{[0,\infty)}$ is a concave function,
then the addition $\underset{T}{\oplus}$ preserves A-similarity of the right parts.

Proof. (i) Let $A \in \mathcal{A}^0$, $B \overset{\delta,\epsilon}{\sim} A$, and $C = A \underset{T}{\oplus} B$. From (5) and the fact $f \circ \mu_A(x) = ax^s$, $x > 0$, we get

$$f \circ \mu_C(z) = \inf_{[0,z]} \left(ax^s + a\epsilon^s(z - x)^s\right), \quad z > 0. \tag{13}$$

Put $h(x) = ax^s + a\epsilon^s(z - x)^s$, $x \in [0, z]$. The function h is differentiable, and its derivative is given by :

$$h'(x) = asx^{s-1} + a\epsilon^s s(z - x)^{s-1}, \quad x \in (0, z).$$

The function h is for $s > 1$ strictly convex, and therefore

$$\inf_{x \in [0,z]} h(x) = h(x_0),$$

where x_0 is the point satisfying $h'(x_0) = 0$. It is only a matter of computation to show that

$$x_0 = \frac{z}{1 + \epsilon^{-\frac{s}{s-1}}}.$$

If we substitute $h(x_0)$ to (13), we get

(14) $$f \circ \mu_C(z) = a \frac{z^s}{\left(1 + \epsilon^{-\frac{s}{s-1}}\right)^{s-1}}, \quad z > 0.$$

Denote

$$\frac{1}{\left(1 + \epsilon^{-\frac{s}{s-1}}\right)^{s-1}} = \gamma^s.$$

Then (14) can be written in the form :

(15) $$f \circ \mu_C(z) = a(\gamma z)^s, \quad z > 0,$$

where γ is a positive number satisfying the condition

(16) $$\gamma^{-\frac{s}{s-1}} = 1 + \epsilon^{-\frac{s}{s-1}}, \quad s > 1.$$

From (15), it follows that

$$\mu_C(z) = f^{-1}\left(a(\gamma z)^s\right) = \mu_A(\gamma z), \quad z > 0,$$

which, by Def.1, means that right parts of C and A are similar and the coefficient of similarity equals to γ. This fact will be denoted by $C_R \overset{\gamma}{\sim} A_R$.
The case $s = 1$ need not be discussed here since it is also included in (ii).

(ii) If the function $f \circ \mu_A|_{[0,\infty)}$ is concave, then, due to similarity $B \sim A$, $f \circ \mu_B|_{[0,\infty)}$ is concave, too. By [6], for the sum $A \underset{T}{\oplus} B$ in the case of concave membership functions, it holds

$$\mu_{A \underset{T}{\oplus} B}(z) = \max\left(\mu_A(z), \mu_B(z)\right).$$

The assumption $B \overset{\delta,\epsilon}{\sim} A$ implies for $z > 0$ the equality $\mu_B(z) = \mu_A(\epsilon z)$, and therefore, for $z > 0$

(17) $$\mu_{A \underset{T}{\oplus} B}(z) = \mu_A(\omega z), \quad \text{where } \omega = \min(1, \epsilon).$$

The equality (17) means that the addition $\underset{T}{\oplus}$ preserves A-similarity of the right parts and $(A \underset{T}{\oplus} B)_R \overset{\omega}{\sim} A_R$.

□

During the proof of the previous theorem we have also obtained the results on the cofficients of similarity. We stress it in the following assertion.

COROLLARY 3. *Let T be a strict t-norm with an additive generator f. Let $A \in \mathcal{A}^0$, $\mu_A(x) > 0$ for $x \geq 0$, and let $B \in \mathcal{A}^0$, $B_R \overset{\epsilon}{\sim} A_R$.*

(i) If $\mu_A(x) = f^{-1}(ax^s)$ for $x \in (0, \infty)$ and any $a > 0$, $s \geq 1$, then

$$(A \underset{T}{\oplus} B)_R \overset{\gamma}{\sim} A_R, \quad \text{where} \quad \gamma^{-\frac{s}{s-1}} = 1 + \epsilon^{-\frac{s}{s-1}}.$$

(ii) If $f \circ \mu_A|_{[0,\infty)}$ is concave, then

$$(A \underset{T}{\oplus} B)_R \overset{\omega}{\sim} A_R, \quad \text{where} \quad \omega = \min(1, \epsilon).$$

The analogous properties can be formulated for left sides.

EXAMPLE 2. Let $T = T_p$, where T_p is a product t-norm given by

$$T_p(x, y) = xy, \quad x, y \in [0, 1].$$

The t-norm T_p is a strict t-norm with an additive generator f, $f \colon [0,1] \to [0, \infty]$, $f(x) = -log x$. Let $A \in \mathcal{A}^0$ and $\mu_A(x) > 0$ for each $x \geq 0$. Let $f \circ \mu_A|_{[0,\infty)}$ be a convex function. By Theorem 1, the necessary condition for preserving of similarity is

$$f \circ \mu_A(x) = -log \mu_A(x) = ax^s$$

for some $a > 0$, $s \geq 1$, and all $x \in (0, \infty)$. It means that μ_A has to be of the form

$$\mu_A(x) = e^{-ax^s}. \tag{18}$$

If μ_A is in the interval $(0, \infty)$ given by (18), then, for each $a > 0$, $s \geq 1$, by Theorem 2, (i), the sufficient condition for preserving of A-similarity is fulfilled.

Give a numerical example (generalized for both sides). Consider A, $B \in \mathcal{A}^0$,

$$\mu_A(x) = e^{-x^2}, \quad x \in \mathbb{R} \quad \text{and} \quad \mu_B(x) = e^{-4x^2}, \quad x \in \mathbb{R}.$$

It is evident that $B \overset{2,2}{\sim} A$ and $s = 2$. By Corollary 3 (i),

$$C = A \underset{T_p}{\oplus} B \overset{\gamma,\gamma}{\sim} A, \quad \text{where} \quad \gamma^{-2} = 1 + 2^{-2} = \frac{5}{4}, \quad i.e., \ \gamma = \frac{2}{\sqrt{5}}.$$

Therefore

$$\mu_C(x) = \mu_A\left(\frac{2}{\sqrt{5}}x\right) = e^{-\frac{4}{5}x^2}, \quad x \in \mathbb{R}.$$

The t-norms in Theorems 1 and 2 were assumed to be strict. The analogous assertions can also be formulated for nilpotent t-norms. An additive generator f of a nilpotent t-norm is a bounded function, i.e., $f \colon [0,1] \to [0, M]$, $M \in \mathbb{R}$. As it was mentioned above, the function $g = f \circ \mu_A|_{[0,\infty)}$ is continuous, non-decreasing, and $g(0) = 0$. If f is bounded, then g is bounded,too. Since there is no such function g which would have the named properties and would also be convex, it has no sense to formulate the conditions of the type (i) given in Theorems 1 and 2.

THEOREM 3. *Let $A \in \mathcal{A}^0$ and $\mu_A(x) > 0$ for each $x \geq 0$. Let T be a nilpotent t-norm with an additive generator f.*

(i) If the addition $\underset{T}{\oplus}$ preserves A-similarity, and $f \circ \mu_A|_{[0\infty)}$ is a concave function, then

$$\mu_{A \underset{T}{\oplus} A}(z) = \mu_A(z), \quad z > 0.$$

(ii) If $f \circ \mu_A|_{[0,\infty)}$ is a concave function, then the addition $\underset{T}{\oplus}$ preserves A-similarity of the right parts.
Moreover, if $B_R \overset{\epsilon}{\sim} A_R$, then

$$(A \underset{T}{\oplus} B)_R \overset{\omega}{\sim} A_R, \quad \omega = \min(1, \epsilon).$$

EXAMPLE 3. Consider the t-norm T_L given by

$$T_L(x, y) = \max(x + y - 1, 0), \quad x, y \in [0, 1].$$

The t-norm T_L (so-called Lukasiewicz t-norm) is nilpotent, with the normed additive generator f, $f\colon [0,1] \to [0,1]$, $f(x) = 1 - x$.
Consider any fuzzy number A with a membership function μ_A, which is convex in the interval $[0, \infty)$. It means that the function $f \circ \mu_A = 1 - \mu_A$ is in the interval $[0, \infty)$ concave. By Theorem 3, (ii), the addition $\underset{T}{\oplus}$ preserves A-similarity of the right parts.

Again, give a generalized numerical example. Consider A, $B \in \mathcal{A}$,

$$\mu_A(x) = \begin{cases} e^{\frac{x}{3}}, & \text{for } x \leq 0, \\ e^{-2x}, & \text{for } x \geq 0 \end{cases}$$

and

$$\mu_B(x) = e^{-|x-4|}, \quad x \in \mathbb{R}.$$

Let $B^0 \in \mathcal{A}^0$, $\mu_{B^0}(x) = e^{-|x|}$, $x \in \mathbb{R}$. It is evident that $B^0 \overset{1,1}{\sim} B$, and $B_0 \overset{3,0.5}{\sim} A$. By Theorem 3, (ii),

$$A \underset{T_L}{\oplus} B^0 \overset{\omega_1,\omega_2}{\sim} A, \quad \text{where } \omega_1 = \min(1, 3) = 1 \text{ and } \omega_2 = \min(1, 0.5) = 0.5.$$

Using $A \underset{T_L}{\oplus} B \in \mathcal{A}^4$, $A \underset{T_L}{\oplus} B \overset{1,1}{\sim} A \underset{T_L}{\oplus} B^0 \overset{1,0.5}{\sim} A$, we get

$$\mu_{A \underset{T_L}{\oplus} B}(x) = \begin{cases} \mu_A(x-4) = e^{-\frac{x-4}{3}}, & \text{for } x \leq 4, \\ \mu_A\left(\frac{1}{2}(x-4)\right) = e^{-(x-4)}, & \text{for } x \geq 4. \end{cases}$$

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