Fuzzy Bimodules Over Fuzzy Subrings

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Abstract: In this paper, we introduce the concept of fuzzy bimodule over fuzzy subrings and discuss its properties. If $f:M \to N$ is a homomorphic epimorphism of modules, we establish a one-to-one order preserving correspondence between the fuzzy bimodules of N and those of M which are constant on the Kerf of f.
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1. Fuzzy bimodule

Let X be a nonempty set, a fuzzy subset u_X of X is a mapping $u_X: X \to [0,1]$. The set $(u_X)_\lambda = \{x \in X: u_X(x) \geq \lambda\}$ is called a level subset of u_X with respect to λ, where $\lambda \in [0,1]$.

Unless specially state, M in this paper refers to $\langle R, S\rangle$-bimodule, where R and S stand for rings with unit element, u_R and u_S denote fuzzy subrings of R and S, respectively.

Definition 1.1 A fuzzy subset u_M of $\langle R,S\rangle$-bimodule M is called a fuzzy bimodule over fuzzy subring $\langle u_R, u_S\rangle$, if for all $x,y \in M, r \in R, s \in S$,

(1) $u_M(x+y) \geq u_M(x) \wedge u_M(y)$

(2) $u_M(rxs) \geq u_R(r) \wedge u_S(s) \wedge u_M(x)$

In brief, u_M is a $\langle u_R, u_S\rangle$-fuzzy bimodule of M.

Proposition 1.2 Let u_M be a $\langle u_R, u_S\rangle$-fuzzy bimodule, then

(1) $u_M(-x) = u_M(x) \quad \forall x \in M$

(2) $u_M(x-y) \geq u_M(x) \wedge u_M(y) \quad \forall x, y \in M$

Proposition 1.3 Let u_M be a fuzzy subset of M, then u_M is a $\langle u_R, u_S\rangle$-fuzzy bimodule of M iff for all $\lambda \in [0, 1]$, $(u_M)_\lambda$ is an addition subgroup. In particular, if $(u_R)_\lambda$ and $(u_S)_\lambda$ are nonempty, then $(u_M)_\lambda$ is a $\langle (u_R)_\lambda, (u_S)_\lambda\rangle$-bimodule.

This proposition can be obtained by [1].

Definition 1.4 Let A and B be fuzzy subsets of M, for $\forall r \in R$, $\forall s \in S$, define fuzzy subsets of M:

$(A\bigcap B)(x) = A(x)\bigwedge B(x)$

$(A+B)(x) = \sup\{A(x_1)\bigwedge B(x_2): x_1+x_2=x\}$

$(rAs)(x) = \sup\{u_R(r)\bigwedge u_S(s)\bigwedge A(x_1): rx_1s = x\}$

Proposition 1.5 Let $\{u_M^k: k\in I\}$ be a family of $\langle u_R,u_S\rangle$-fuzzy bimodules of M, where I denotes index set, then $\bigcap\{u_M^k: k\in I\}$ is a $\langle u_R,u_S\rangle$-fuzzy bimodule of M.

The proposition is straightforward.

Proposition 1.6 A fuzzy subset u_M of M is a $\langle u_R, u_S\rangle$-fuzzy bimodule iff

(1) $u_M + u_M\leqslant u_M$

(2) $\forall r \in R$, $\forall s \in S$, $ru_Ms\leqslant u_M$

Proof. If u_M is a $\langle u_R, u_S\rangle$-fuzzy bimodule of M, then for $\forall x\in M$, $\forall r\in R$, $\forall s\in S$, we have

$$(u_M+u_M)(x) = \sup\{u_M(x_1)\bigwedge u_M(x_2): x_1+x_2=x\}$$
$$\leqslant \sup\{u_M(x_1+x_2): x_1+x_2=x\} = u_M(x)$$

i.e. $u_M+u_M\leqslant u_M$

$$(ru_Ms)(x) = \sup\{u_R(r)\bigwedge u_S(s)\bigwedge u_M(x_1): rx_1s=x\}$$
$$\leqslant \sup\{u_M(rx_1s): rx_1s=x\} = u_M(x)$$

i.e. $ru_Ms\leqslant u_M$

Inversely, for $\forall x,y\in M$, $\forall r\in R$, $\forall s\in S$,

From $u_M + u_M\leqslant u_M$ and $ru_Ms\leqslant u_M$, we have

$$u_M(x+y)\geqslant(u_M+u_M)(x+y)$$
$$= \sup\{u_M(x_1)\bigwedge u_M(x_2): x_1+x_2=x+y\} \geqslant u_M(x)\bigwedge u_M(y)$$

$$u_M(rxs) \geqslant (ru_Ms)(rxs) = \sup\{u_R(r)\bigwedge u_S(s)\bigwedge u_M(x_1): rx_1s=rxs\}$$

$\geqslant u_R(r) \wedge u_S(s) \wedge u_M(x)$

Hence, u_M is a $\langle u_R, u_S\rangle$-fuzzy bimodule.

Proposition 1.7 Let u_M and v_M be $\langle u_R, u_S\rangle$-fuzzy bimodules of M, then u_M+v_M is a $\langle u_R, u_S\rangle$-fuzzy bimodule of M.

Proof. $\forall x, y \in M$, $\forall r \in R$, $\forall s \in S$,

$(u_M+v_M)(x+y) \geqslant \sup\{u_M(x_1+y_1) \wedge v_M(x_2+y_2) : x_1+x_2=x, y_1+y_2=y\}$

$\geqslant \sup\{u_M(x_1) \wedge v_M(x_2) \wedge u_M(y_1) \wedge v_M(y_2) : x_1+x_2=x, y_1+y_2=y\}$

$= \sup\{u_M(x_1) \wedge v_M(x_2) : x_1+x_2=x\} \wedge \sup\{u_M(y_1) \wedge v_M(y_2) : y_1+y_2=y\}$

$= (u_M+v_M)(x) \wedge (u_M+v_M)(y)$

$(u_M+v_M)(rxs) \geqslant \sup\{u_M(rx_1s) \wedge v_M(rx_2s) : x_1+x_2=x\}$

$\geqslant \sup\{u_R(r) \wedge u_S(s) \wedge u_M(x_1) \wedge v_M(x_2) : x_1+x_2=x\}$

$= u_R(r) \wedge u_R(s) \wedge \sup\{u_M(x_1) \wedge v_M(x_2) : x_1+x_2=x\}$

$= u_R(r) \wedge u_S(s) \wedge (u_M+v_M)(x)$

Hence, u_M+v_M is a $\langle u_R, u_S\rangle$-fuzzy bimodule of M.

Proposition 1.8 Let R and S be commutative rings, M a $\langle R,S\rangle$-bimodule, u_R and u_S fuzzy subrings of R and S, respectively. If u_M is a $\langle u_R, u_S\rangle$-fuzzy bimodule of M, then for $\forall r \in R, \forall s \in S$, ru_Ms is a $\langle u_R, u_S\rangle$-fuzzy bimodule of M.

Proof. $\forall x, y \in M, \forall a \in R, \forall b \in S$

$(ru_Ms)(x+y) = \sup\{u_M(x_1+x_2) : r(x_1+x_2)s=x+y\}$

$\geqslant \sup\{u_M(x_1) \wedge u_M(x_2) : rx_1s=x, rx_2s=y\}$

$= \sup\{u_M(x_1) : rx_1s=x\} \wedge \sup\{u_M(x_2) : rx_2s=y\}$

$= (ru_Ms)(x) \wedge (ru_Ms)(y)$

$(ru_Ms)(axb) \geqslant \sup\{u_M(ax_1b) : r(ax_1b)s = axb\}$

$\geqslant \sup\{u_R(a) \wedge u_S(b) \wedge u_M(x_1) : rx_1s=x\}$

$= u_R(a) \wedge u_S(b) \wedge \sup\{u_M(x_1) : rx_1s=x\}$

$= u_R(a) \wedge u_S(b) \wedge (ru_Ms)(x)$

So, ru_Ms is a $\langle u_R, u_S\rangle$-fuzzy bimodule of M.

Proposition 1.9 Let R and S be commutative rings, M a $\langle R,S\rangle$-bimodule, u_R and u_S be fuzzy subrings of R and S, respectively. If u_M and v_M are $\langle u_R, u_S\rangle$-fuzzy bimodules of M, then for $\forall r \in R, \forall s \in S$

$$ru_Ms+rv_Ms = r(u_M+v_M)s$$

Proof. $\forall x\in M$, let $(ru_Ms+rv_Ms)(x)=\alpha$

i.e. $\sup\{(ru_Ms)(x_1)\wedge(ru_Ms)(x_2): x_1+x_2=x\}=\alpha$

then for $\forall\varepsilon>0$, $\exists\bar{x}_1, \bar{x}_2\in M$, such that $\bar{x}_1+\bar{x}_2=x$ and

$(ru_Ms)(\bar{x}_1)\wedge(ru_Ms)(\bar{x}_2)>\alpha-\varepsilon/2$

Also, by definition of ru_Ms, $\exists\bar{a},\bar{b}\in M$, such that $r\bar{a}s=\bar{x}_1$, $r\bar{b}s=\bar{x}_2$ and

$u_M(\bar{a}) > (ru_Ms)(\bar{x}_1)-\varepsilon/2 >\alpha-\varepsilon$

$v_M(\bar{b}) > (rv_Ms)(\bar{x}_2)-\varepsilon/2 >\alpha-\varepsilon$

So, $[r(u_M+v_M)s](x)=\sup\{\sup[u_M(p)\wedge v_M(q): p+q=x_1]: rx_1s=x\}$

$\geqslant u_M(\bar{a})\wedge v_M(\bar{b}) >\alpha-\varepsilon$

Hence, $[r(u_M+v_M)s](x)\geqslant(ru_Ms+rv_Ms)(x)$, since ε is arbitrary.

Inversely, let $[r(u_M+v_M)s](x)=\beta$

i.e. $\sup\{(u_M+v_M)(x_1): rx_1s=x\}=\beta$

then for $\forall\varepsilon>0$, $\exists\bar{x}_1\in M$, such that $r\bar{x}_1s=x$ and

$(u_M+v_M)(\bar{x}_1)>\beta-\varepsilon/2$

Also, by defition of u_M+v_M, $\exists\bar{p},\bar{q}\in M$, such that $\bar{p}+\bar{q}=\bar{x}_1$ and

$u_M(\bar{p})\wedge v_M(\bar{q}) > (u_M+v_M)(\bar{x}_1)-\varepsilon/2 >\beta-\varepsilon$

So, $(ru_Ms+rv_Ms)(x)$

$=\sup\{\sup[u_M(p): rps=x_1]\wedge\sup[v_M(q): rqs=x_2]: x_1+x_2=x\}$

$\geqslant u_M(\bar{p})\wedge v_M(\bar{q}) >\beta-\varepsilon$

Hence, $(ru_Ms+rv_Ms)(x)\geqslant[r(u_M+v_M)s](x)$, since ε is arbitrary.

i.e. $(ru_Ms+rv_Ms)(x)=[r(u_M+v_M)s](x)$

That is, $ru_Ms + rv_Ms = r(u_M + v_M)s$.

2. Correspondence theorem

Let M and N be two $\langle R,S\rangle$-bimodules, $f: M\to N$ be a homomorphic mapping from M to N, u_R and u_S be fuzzy subrings of R and S, respectively, $F(M)$ and $F(N)$ stand for the sets composed of all $\langle u_R,u_S\rangle$-fuzzy bimodules of M and N, respectively. Let $u_M\in F(M)$, $u_N\in F(N)$, then $f(u_M)$ and $f^{-1}(u_N)$ is defined by

$\forall y\in N, f(u_M)(y) = \begin{cases} \sup\{u_M(x): f(x)=y\} & \text{if } f^{-1}(y)\neq\varphi \\ 0 & \text{otherwise} \end{cases}$

$\forall x\in M, f^{-1}(u_N)(x) = u_N(f(x))$

Proposition 2.1 Let $u_M, v_M\in F(M), u_N, v_N\in F(N)$, then

(1) $u_M\leqslant v_M \Longrightarrow f(u_M)\leqslant f(v_M)$

(2) $u_N\leqslant v_N \Longrightarrow f^{-1}(u_N)\leqslant f^{-1}(v_N)$

(3) $f[f^{-1}(u_N)] = u_N$, if f is an epimorphism.

(4) $f^{-1}[f(u_M)] = u_M$, if u_M is constant on Kerf.

This proposition is obvious.

Proposition 2.2 Let $u_N\in F(N)$, then $f^{-1}(u_N)\in F(M)$ and $f^{-1}(u_N)$ is constant on Kerf.

Proof. It is clear that $f^{-1}(u_N)$ is constant $u_N(0)$ on Kerf. Further, $\forall x_1, x_2\in M$, $\forall r\in R, \forall s\in S$,

$f^{-1}(u_N)(x_1+x_2) = u_N(f(x_1+x_2)) = u_N(f(x_1)+f(x_2))$

$\geqslant u_N(f(x_1))\wedge u_N(f(x_2)) = f^{-1}(u_N)(x_1)\wedge f^{-1}(u_N)(x_2)$

$f^{-1}(u_N)(rx_1s) = u_N(f(rx_1s)) = u_N(rf(x_1)s)$

$\geqslant u_R(r)\wedge u_S(s)\wedge u_N(f(x_1)) = u_R(r)\wedge u_S(s)\wedge f^{-1}(u_N)(x_1)$

Hence, $f^{-1}(u_N)\in F(M)$.

Proposition 2.3 Let $u_M\in F(M)$. If u_M is constant on Kerf, then $f(u_M)\in F(N)$, where f is an epimorphism from M to N.

Proof. $\forall y_1, y_2\in N$, $\forall r\in R, \forall s\in S$,

$f(u_M)(y_1+y_2) = \sup\{u_M(x_1+x_2): f(x_1+x_2)=y_1+y_2\}$

$\geqslant \sup\{u_M(x_1)\wedge u_M(x_2): f(x_1)=y_1, f(x_2)=y_2\}$

$=\sup\{u_M(x_1): f(x_1)=y_1\}\wedge \sup\{u_M(x_2): f(x_2)=y_2\}$

$=f(u_M)(y_1)\wedge f(u_M)(y_2)$

$f(u_M)(ry_1s)\geqslant \sup\{u_M(rx_1s): f(rx_1s)=ry_1s\}$

$\geqslant \sup\{u_R(r)\wedge u_S(s)\wedge u_M(x_1): f(x_1)=y_1\}$

$= u_R(r)\wedge u_S(s)\wedge \sup\{u_M(x_1): f(x_1)=y_1\}$

$= u_R(r)\wedge u_S(s)\wedge f(u_M)(y_1)$

Hence, $f(u_M)\in F(N)$.

From the above discussion, we can draw the following theorem.

Theorem 2.4 Let $f:M\to N$ is an epimorphism, then there is a one-to-one order preserving correspondence between the $\langle u_R, u_S\rangle$-fuzzy bimodules of N and those of M which are constant on Kerf.

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