FUZZY REGULAR AND INVERSE SUBSEMIGROUPS

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ABSTRACT: Regular and inverse $(a,b;\ \in_{\alpha},\in_{\alpha}vq_{(\alpha,b)})$ – fuzzy subsemigroups are defined. The concept of fuzzy analogue of normal subsemigroup is introduced. Using a type of fuzzy congruence relation, a fuzzy quotient subgroup of a fuzzy inverse subsemigroup is constructed. Some results concerning the interrelationship between fuzzy analogues of unitary subsemigroup, inverse subsemigroup and closed subsemigroup of an inverse semigroup are obtained.

KEYWORDS: Fuzzy algebra, fuzzy subsemigroup, fuzzy k-congruence relation, regular and inverse semigroup.

1. INTRODUCTION:

To deal with the problem of chracterization of fuzzy subsystems λ of a universal algebra G such that λ_t are subalgebras of G for all t in a pre-assigned subinterval [a,b] of [0,1], some new types of fuzzy subalgebras were introduced by Das [3] of which $(a,b;\ \in_{\alpha},\in_{\alpha}vq_{(\alpha,b)})$ fuzzy subalgebra was found to be quite interesting. That necessitates thorough investigation of different aspects of such fuzzy subsystems of different algebraic systems.

Rosenfeld- type fuzzy subsemigroups have been discussed by Al- Thukair [1], Kuroki [5 - 7] ,Jizhong Shen [8] and others. In particular, fuzzy regular subsemigroups have been extensively studied by Jizhong Shen [8]. Regular and inverse semigroups are two of the most important classes of semigroups. The object of the present paper is to study $(a,b; \in_\alpha, \in_\alpha \vee q_{(a,b)})$- type fuzzification of these concepts.

Regular $(a,b; \in_\alpha, \in_\alpha \vee q_{(a,b)})$ - fuzzy subsemigroups are defined in Section 3 and some results analogous to those obtained by Jizhong Shen [8] are obtained.. Inverse fuzzy subsemigroups in the new setting are considered in Section 4. An important feature of inverse semigroup is the group congruence and normal subsemigroup by which a quotient group is obtained from an inverse semigroup. The fuzzy analogue of a normal subsemigroup is defined and using a type of fuzzy congruence relation, a fuzzy quotient subgroup of a fuzzy inverse subsemigroup is constructed. The concepts of left and right unitary subsets are fuzzified and some results concerning the interelationship between the fuzzy analogues of unitary, inverse and closed subsemigroups are obtained.

2. PRELIMINARIES:

Let X be a non-empty set and let $I = [0,1]$.

I^X will denote the set of all mappings $\alpha: X \longrightarrow I$.

If $x,y \in I$, $M(x,y)$ will denote the minimum of x and y.

Let $a,b \in I$ be such that $0 < a < b \leq 1$.

Let $c = M(2b,1)$, $d = M(2b,1+a)$ and $k = d/2$.

DEFINITION 2.1: Let $\lambda \in I^X$. If $0 < r \le 1$, $0 \le t < 1$ and $0 < c < d \le 1$, then λ_r, λ_{st}, $\lambda_{c,d}$ are defined by

$$\lambda_r = \{ x \in G;\ \lambda(x) \ge r\},$$

$$\lambda_{st} = \{x \in G;\ \lambda(x) > t\},$$

$$\lambda_{c,d} = \{x \in G;\ c < \lambda(x) \le d\}.$$

DEFINITION 2.2: Let $\lambda \in I^X$. A fuzzy point (x,t) is said to belong to λ with respect to a, denoted by $(x,t) \in_a \lambda$ (resp. be coincident with λ with respect to (a,b), denoted by $(x,t)\ q_{(a,b)}\ \lambda$) if $\lambda(x) \ge \max.\{a,t\}$ (resp. $\lambda(x) + t > d$).

If $(x,t) \in_a \lambda$ or $(x,t)\ q_{(a,b)}\ \lambda$, then we write $(x,t) \in_a \vee\ q_{(a,b)}\ \lambda$.

Let S be a semigroup.

DEFINITION 2.3: [3] A fuzzy subset λ of S is said to be an $(a,b;\ \in_a,\ \in_a \vee q_{(a,b)})$ fuzzy subsemigroup of S if

$$(x,t) \in_a \lambda,\ (y,t_1) \in_a \lambda \Longrightarrow (xy,\ M\,(t,t_1)) \in_a \vee\ q_{(a,b)}\ \lambda$$

for all $x,y \in S$ and for all $t,t_1 \in (a,c]$;

or equivalently

$$\lambda(xy) \ge M\,(\ \lambda(x)\ ,\lambda(y)\ ,k\ ) \text{ for all } x,y \in \lambda_{sa}.$$

In what follows, unless otherwise mentioned, by a fuzzy subsemigroup of S, we shall mean an $(a,b;\ \in_a,\in_a \vee q_{(a,b)})$ - fuzzy subsemigroup of S.

THEOREM 2.4: [3] A fuzzy subset λ of a semigroup S is a fuzzy subsemigroup of S iff λ_t is a subsemigroup of S $\forall\ t \in (a,k]$.

THEOREM 2.5: [3] Let f be a semigroup epimorphism of S onto a semigroup T. Let λ and μ be fuzzy subsemigroups of S and T respectively. Then

(i) $f(\lambda)$ is a fuzzy subsemigroup of T,

(ii) $f^{-1}(\mu)$ is a fuzzy subsemigroup of S.

DEFINITION 2.6[2]: A fuzzy relation ρ on a set X is said to be a fuzzy c-equivalence relation on X if

(i) ρ is reflexive of order c i.e.

$\rho(x,x) = c$ for all $x \in X$ where $c > 0$, a fixed number

(ii) $\rho(x,y) = \rho(y,x)$ for all $x,y \in X$

(iii) $\rho \geq \rho o \rho$.

DEFINITION 2.7 [1] : A fuzzy relation ρ on a semigroup S is said to be fuzzy left (resp. right) compatible if for all $x,y,s \in S$, $\rho(sx,sy)$ (resp. $\rho(xs,ys)$) $\geq \rho(x,y)$.

DEFINITION 2.8 [1] : A fuzzy c-equivalence relation ρ on semigroup S is said to be a fuzzy c-congruence relation on S i

$M (\rho(x,y), \rho(z,w)) \leq \rho(xz,yw)$ for all $x,y,z,w \in S$.

LEMMA 2.9 [1] : A fuzzy c-equivalence relation ρ on a semigroup S is a fuzzy c-congruence relation iff it is both fuzzy left and right compatible .

Unless otherwise stated we follow Howie [4] regarding the notions and notations of semigroups .

3. FUZZY REGULAR SUBSEMIGROUP

Let S,T be regular semigroups . For all $x \in S$ let $R_x = \{ x_1 \in S ; x x_1 x = x \}$.

DEFINITION 3.1 : A fuzzy subsemigroup λ of S is called a fuzzy regular subsemigroup if for all $x \in S$, $\exists\, x_1 \in R_x$ such that $(x,t) \in_\alpha \lambda \Rightarrow (x_1,t) \in_\alpha \vee q_{(a,b)} \lambda$ for all $t \in (a,c]$ or equivalently for all $x \in \lambda_{a\alpha}$, $\exists\, x_1 \in R_x$ such that

$\lambda(x_1) \geq M (\lambda(x) , k)$.

EXAMPLE 3.2 : Let S be a semigroup having the following composition table

	o	e	f	a	b
o	o	o	o	o	o
e	o	e	o	a	o
f	o	o	f	o	b
a	o	o	a	o	e
b	o	b	o	f	o

Clearly S is regular .

$\lambda : S \longrightarrow I$, defined by

$\lambda(o) = .6$, $\lambda(e) = .5$, $\lambda(f) = .45$, $\lambda(a) = \lambda(b) = .4$

is a $(.2,.7\ ;\ \in_{.2},\ \in_{.2} \vee q_{(.2,.7)})$ fuzzy regular subsemigroup of S .

THEOREM 3.3 : A fuzzy subset λ of S is a fuzzy regular subsemigroup of S iff λ_t is a regular subsemigroup of S for all $t \in (a,k]$.

THEOREM 3.4 : Let f be a semigroup epimorphism from S onto T . Let λ and μ be fuzzy regular subsemigroups of S and T respectively . Then

(i) $f(\lambda)$ is a fuzzy regular subsemigroup of T .

(ii) $f^{-1}(\mu)$ is a fuzzy regular subsemigroup of S .

THEOREM 3.5 : If μ is a fuzzy regular subsemigroup of S , then $\mu.\mu|P = \mu|P$ where $P = \mu_{sa} - \mu_k$

and $(\mu.\mu)(x) = \sup \{ M(\mu(y),\mu(z))\ ;\ x = yz \}$ for all $x \in S$.

Let $S^1 = S \cup \{1\}$ where $x.1 = x = 1.x$ for all $x \in S$. Now for each fuzzy subset μ of S , we define a fuzzy subset μ^1 of S^1 by $\mu^1(x) = \mu(x)$ for all $x \in S$

$= k$ if $x = 1$.

THEOREM 3.6 : Let λ be a fuzzy subsemigroup of S . Then λ is fuzzy regular iff for all $x \in \lambda_{sa}$, $\exists$ an idempotent element e in λ_t such that $x\lambda^1{}_t = e\lambda_t$ where $t = \lambda(x)$ if $x \in \lambda_{a,k}$ and $t = k$ otherwise .

DEFINITION 3.7 : A fuzzy subsemigroup λ of S is called a weakly fuzzy regular subsemigroup if for all $x \in \lambda_{sa}$ $\sup \{ \lambda(x_1) ; x_1 \in R_x \} \geq M (\lambda(x) ,k)$.

THEOREM 3.8 : A fuzzy subset λ of S is weakly fuzzy regular subsemigroup iff λ_{st} is a regular subsemigroup of S for all $t \in [a,k)$.

REMARK 3.9 : Theorem 3.4 is true if ,` fuzzy regular´ is replaced by ` weakly fuzzy regular ´ .

THEOREM 3.10 : Let μ be a fuzzy subsemigroup of S with the condition that for all $x \in \mu_{sa}$, $x_1 \in \mu_{sa}$ for all $x_1 \in R_x$ .Then μ is weakly fuzzy regular iff $\sup \{ \mu(x_1 x x_1) ; x_1 \in R_x \} \geq M (\mu(x) ,k)$.

DEFINITION 3.11 : Let $\mu \in I^S$ and let $x,y,z \in S$. Let $P = \{ s \in S ; xsy = z \}$. Then $x\mu y : S \longrightarrow I$ is defined by $x\mu y(z) = \emptyset$ or $\sup \{ \mu(s) ; s \in P \}$ according as $P = \phi$ or $P \neq \phi$

THEOREM 3.12 : A fuzzy subsemigroup μ of S is weakly fuzzy regular iff for all $x \in \lambda_{sa}$, $(x\mu x)(x) \geq M (\mu(x) ,k)$.

4. FUZZY INVERSE SUBSEMIGROUP

Let S be an inverse semigroup .

DEFINITION 4.1 : A fuzzy subsemigroup λ of S is called a fuzzy inverse subsemigroup if

$(x,t) \in_a \lambda \Rightarrow (x^{-1},t) \in_a \vee\, q_{(a,b)}\, \lambda$ for all $x \in S$, for all $t \in (a,c]$

or equivalently

$\lambda(x^{-1}) \geq M (\lambda(x) ,k)$ for all $x \in \lambda_{sa}$.

EXAMPLE 4.2 : Let M(P) be the set of all partial one-one mappings on the set P = {a,b} . Then M(P) = { A , B , C , D , E , F , ϕ } where A = { (a,a) } , B = { (a,b) } , C = { (b,b)} , D = { (b,a) } , E = { (a,a),(b,b) } , F = { (a,b),(b,a) } and ϕ is the empty relation . Define binary operation on M(P) as the usual composition of binary relations .

M(P) is an inverse semigroup with $A^{-1} = A$, $B^{-1} = D$, $C^{-1} = C$, $E^{-1} = E$, $F^{-1} = F$, $\phi^{-1} = \phi$. Define a fuzzy subset λ of M(P) by $\lambda(A) = .55$, $\lambda(B) = \lambda(D) = .5$, $\lambda(C) = .7$, $\lambda(E) = .45$, $\lambda(F) = .1$, $\lambda(\phi) = .6$. It can be easily verified that λ is a (.2 , .7 , $\in_{.2}$, $\in_{.2} \vee q_{(.2,.7)}$) fuzzy inverse subsemigroup of M(P) .

REMARK 4.3 : Theorems 3.3 and 3.4 are valid if ` fuzzy regular ' is replaced by ` fuzzy inverse .

DEFINITION 4.4 : A fuzzy subset λ of S is said to be fuzzy closed , if for all $x \in S$ and for all idempotents e of S , $\lambda(x) \geq M(\lambda(ex), k)$ provided $ex \in \lambda_{s\alpha}$.

THEOREM 4.5 : A fuzzy subset λ of S is fuzzy closed iff λ_t is closed subset of S for all $t \in (a,k]$.

DEFINITION 4.6 : For all $\lambda \in I^S$, $\lambda\omega : S \longrightarrow I$ defined by $(\lambda\omega)(x) = \sup \{M(\lambda(ex), k) ; e$ is an idempotent of S $\}$ for all $x \in S$ is called the fuzzy closure of λ .

THEOREM 4.7 : A fuzzy subset λ of S is fuzzy closed iff $x \in \lambda_{s\alpha} \Longleftrightarrow x \in (\lambda\omega)_{s\alpha}$ and when $\lambda(x) \leq k$, $\lambda(x) = (\lambda\omega)(x)$.

THEOREM 4.8 : For any fuzzy subset λ of S, $(\lambda\omega)_{st} = (\lambda_{st})\omega$ for $t \in [0,k)$.

DEFINITION 4.9 : An fuzzy inverse subsemigroup λ of S is called a fuzzy normal subsemigroup if

(i) $\lambda(e) \geq k$, for all idempotents e of S .

(ii) λ is fuzzy closed .

(iii) $(x,t) \in_{\alpha} \lambda \Rightarrow (yxy^{-1}, t) \in_{\alpha} \vee q_{(\alpha,b)} \lambda$ for all x, y $\in$ S and for all $t \in (a,c]$.

The condition (iii) is equivalent to the condition

(III) $\lambda(yxy^{-1}) \geq M(\lambda(x), k)$ for all $x \in \lambda_{s\alpha}$, for all $y \in S$.

THEOREM 4.10 : A fuzzy subset λ of S is fuzzy normal subsemigroup of S iff λ_t is normal subsemigroup of S for all $t \in (a,k]$.

THEOREM 4.11 : Let f be a semigroup homomorphism from S to T . If μ is a fuzzy normal subsemigroup of T , then $f^{-1}(\mu)$ is a fuzzy normal subsemigroup of S .

THEOREM 4.12 : Let λ be a fuzzy closed and fuzzy inverse subsemigroup of S such that $\lambda(e) \geq k$, for all idempotent e of S . λ is fuzzy normal iff for all $x,y \in S$

(i) $\lambda(xy) \geq k \Longleftrightarrow \lambda(yx) \geq k$

(ii) $xy \in \lambda_{\alpha,k} \Longleftrightarrow yx \in \lambda_{\alpha,k}$ and also $\lambda(xy) = \lambda(yx)$.

THEOREM 4.13 : Let λ be a fuzzy normal subsemigroup of S with the condition that $a = \emptyset$. The fuzzy relation ρ on S defined by $\rho(x,y) = M(\lambda(xy^{-1}), k)$ for all $x,y \in S$ is a fuzzy k-congruence relation on S, called the fuzzy congruence relation induced by λ .

THEOREM 4.14 : $S/\rho = \{ \rho_x ; x \in S \}$ is a group with respect to the binary operation defined by $\rho_x . \rho_y = \rho_{xy}$. $\bar{\lambda} : S/\rho \longrightarrow I$ defined by $\bar{\lambda}(\rho_x) = \lambda(x)$ is a fuzzy subgroup of S/ρ , called the fuzzy quotient subgroup of S determined by λ .

THEOREM 4.15 : Let ρ be a fuzzy k-congruence relation on an inverse semigroup S such that S/ρ is a group under the composition $\rho_x.\rho_y = \rho_{xy}$ where ρ_x is defined by $\rho_x(y) = \rho(x,y)$ for all $y \in S$. Then $\exists$ a fuzzy normal subsemigroup μ of S (with a = $\emptyset$) such that $\rho_\mu |T = \rho|T$ where ρ_μ is the fuzzy k-congruence relation induced by μ and $T = \{ (x,y) \in SxS ; \rho(x,y) \leq k \}$.

DEFINITION 4.16 : A fuzzy subsemigroup λ of S is called

(i) fuzzy right unitary if

$(y,t) \in_\alpha \lambda,\ (xy,t_1) \in_\alpha \lambda \Longrightarrow (x,M\ (t,t_1)) \in_\alpha \vee\, q_{(\alpha,b)}\ \lambda$

for all $x,y \in S$ and for all $t,t_1 \in (a,c]$

or equivalently

$\lambda(x) \geq M\ (\ \lambda(y)\ ,\lambda(xy)\ ,k\ )$ for all $y,xy \in \lambda_{s\alpha}$;

(ii) fuzzy left unitary if

$(x,t) \in_\alpha \lambda,\ (xy,t_1) \in_\alpha \lambda \Longrightarrow (y,M\ (t,t_1)) \in_\alpha \vee\, q_{(\alpha,b)}\ \lambda$

for all $x,y \in S$ and for all $t,t_1 \in (a,c]$

or equivalently

$\lambda(y) \geq M\ (\ \lambda(x)\ ,\lambda(xy)\ ,k\ )$ for all $x,xy \in \lambda_{s\alpha}$.

A fuzzy subsemigroup λ of S is called fuzzy unitary if it is both fuzzy right and left unitary.

THEOREM 4.17 : A fuzzy subset λ of S is a fuzzy unitary subsemigroup of S iff λ_t is a unitary subsemigroup of S for all $t \in (a,k]$.

THEOREM 4.18 : Let λ be a fuzzy subsemigroup of S.

(i) If λ is fuzzy left unitary and $\lambda(e) \geq k$ for all idempotents e of S, then λ is a fuzzy inverse subsemigroup;

(ii) If λ is a fuzzy left unitary inverse subsemigroup, then λ is fuzzy unitary;

(iii) A fuzzy inverse subsemigroup λ is fuzzy unitary iff it is fuzzy closed.

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