ROUGH METRIC SPACES

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Abstract :

In this paper the author introduces the concept of rough metric spaces, and studies some propositions.

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Rough sets, rough metrics, rough distance functions, rough distances, rough metric spaces, rough diameter, rough open balls, rough interior points, rough open sets.

1. Introduction

Pawlak [4,8] introduced the concept of rough sets. Since then it has become an attractive area of research in different fields. Algebraic approach to rough sets was studied by Iwinski [3]. In [1], Dubois and Prade studied rough fuzzy sets and fuzzy rough sets. They also suggested some research directions in the same paper [1].

In the present paper the author introduces the concept of rough metric spaces and studies several properties and propositions. The rough metric spaces are not metric spaces in general; but metric spaces can be regarded as rough metric spaces.

2. Preliminaries

We give below some preliminaries on rough sets.

Definition 2.1

Let U be a non-empty set and $\mathbb{B}$ be a complete algebra of the Boolean algebra $\mathbb{P}(U)$ of subsets of U. The pair $(U,\mathbb{B})$ is called a rough universe.

Definition 2.2

Let $V = (U,\mathbb{B})$ be a given fixed rough universe. Let R be a relation defined as follows :

$A = (\underline{A}, \bar{A}) \in R$ iff $\underline{A}, \bar{A} \in \mathbb{B}$ and $\underline{A} \subseteq \bar{A}$.

Then the elements of R are called rough sets, and the elements of $\mathbb{B}$ are called exact sets.

We see that (X,X) is a rough set $\forall$ X $\in$ $\mathbb{B}$. Thus in this sense of idnetification an axact set can be viewed as a rough set. But a rough set is not an exact set in general. S = (U,R) is called here the approximation space.

Definition 2.2

Let S = (U,R) be an approximation space. Suppose X $\subseteq$ U, where X is not a null set. Then the sets

$\underline{A}(X) = \{x : [x]_R \subseteq X\}$ and

$\bar{A}(X) = \{x : [x]_R \cap X \neq \phi\}$

are called respectively lower and upper approximations of the set X in the approximation space S, where $[x]_R$ denotes the equivalence class of the relation R containing x. The rough set $A(X) = (\underline{A}(X), \bar{A}(X))$ is called the rough set of X in S. For a fixed approximation space S = (U,R) and for a fixed non-null subset X of U, the rough set of X i.e. A(X) is unique.

Definition 2.3

Let $A = (\underline{A}, \bar{A})$ and $B = (\underline{B}, \bar{B})$ be any two rough sets in the approximation space S = (U,R). Then

(i) $A \cup B = (\underline{A} \cup \underline{B}, \bar{A} \cup \bar{B})$

(ii) $A \cap B = (\underline{A} \cap \underline{B}, \bar{A} \cap \bar{B})$

(iii) $A \subset B$ iff $A \cap B = A$.

We say that A is a rough subset of B or B is a rough superset of A. Thus $A \subset B$ iff $\underline{A} \subset \underline{B}$ and $\bar{A} \subset \bar{B}$. This property of rough inclusion has all the properties of set inclusion.

(iv) The natural inverse rough set of A denoted by -A is defined by

$$- A = (U - \bar{A},\ U - \underline{A})$$

This -A is also called rough complement of A in (U,R).

(v) $A - B = A \cap (-B) = (\underline{A} - \bar{B},\ \bar{A} - \underline{B})$

3. Rough Metric Space

Let us define now a rough metric space.

Definition 3.1

Let X be a non-null subset of U and R be an equivalence relation defined on U. Let A(X) be the rough set of X in the approximation space (U,R). Then the function

$$d : X \times X \longrightarrow \mathbb{R}$$

is called a rough metric on X if the following are true:

$\forall\ x,y,z \in X$,

(i) $d(x,y) \geq 0$

$d(x,y) = 0$ iff $[x]_R = [y]_R$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,y) + d(y,z) \geq d(x,z)$

We say that d is a rough metric or rough distance function on X, and $\langle A(X), d \rangle$ is a rough metric space. Rough distance between x and y is d(x,y).

Thus, rough metric spaces are not metric spaces in general. But a metric space is a rough metrix space if the equivalence relation R is such that x is related to y i.e. xRy iff x = y, when $x,y \in X$.

Example 3.1

Let X be a non-empty subset of U. Let A(X) be the rough set of X in the approximation space S = (A,R) where R is an equivalence relation defined on U. Define d by

$$d(x,y) = 0, \quad \text{if } [x]_R = [y]_R$$
$$= 1, \quad \text{otherwise},$$
$$\forall\, x,y \in X.$$

Then d is a rough metric on X and $\langle A(X),d\rangle$ is a rough metric space in the approximation space S.

Proposition 3.1

If $\langle A(X),d\rangle$ is a rough metric space in S = (U,R), then $\langle A(X),d_1\rangle$ is also so in S, where

$$d_1(x,y) = \frac{d(x,y)}{1 + d(x,y)} \quad \forall\, x,y \in X.$$

Proof: Clearly $d_1(x,y) \geq 0$. Also $d_1(x,y) = 0$ iff $[x]_R = [y]_R$, and $d_1(x,y) = d_1(y,x)$. The triangular inequality $d_1(x,z) \leq d_1(x,y) + d_1(y,z)$ can be also proved by little calculation.

Proposition 3.2

If $\langle A(X),d\rangle$ is a rough metric space, then $(X/R,\rho)$ is a metric space where ρ is defined by $\rho([x],[y]) = d(x,y)$.

Proof: Clearly $\rho([x],[y]) \geq 0 \quad \forall\, [x],[y] \in X/R$.

If $\rho([x],[y]) = 0$, then $d(x,y) = 0$ which implies that $[x] = [y]$, and conversely. That $\rho([x],[y]) = \rho([y],[x])$ is obvious. For the triangular inequality we see that

$$\rho([x],[z]) = d(x,z)$$
$$\leq d(x,y) + d(y,z)$$
$$= \rho([x],[y]) + \rho([y],[z])$$

$\forall$ $[x],[y],[z] \in X/R$. Hence proved.

The following propositions are also true. The author states those without proofs.

Proposition 3.3

If $\langle A(X),d\rangle$ is a rough metric space, then $\langle A(X),\rho\rangle$ is also a rough metric space where $\rho(x,y) = \min\{d(x,y),1\}$.

Proposition 3.4

If $\langle A(X),d\rangle$ is a rough metric space, then $\forall$ $x,y,z \in X$,

$$|d(x,z) - d(y,z)| \leq d(x,y)$$

and $\forall$ $x,y,x_1,y_1 \in X$,

$$|d(x,y) - d(x_1,y_1)| \leq |d(x,x_1) + d(y,y_1)|.$$

Definition 3.2

If $\langle A(X),d\rangle$ is a rough metric space, then the rough diameter of the set X is δ given by

$$\delta = \sup_{x,y \in X} d(x,y)$$

4. Rough Open Sets in a Rough Metric Space

We introduce here the concept of rough openness.

Definition 4.1

Given a rough metric space $<A(X),d>$ and a real number $r>0$, a rough open ball $B_r(x)$ of rough radius r about a point $x \in X$ is defined as

$$B_r(x) = \{y \in X \; ; \; d(x,y) < r\}.$$

Thus $B_r(x)$ constains all points of X whose rough distance from x is less than r. Clearly $B_r(x) \neq \phi$. If $r < s$ then $B_r(x) \subseteq B_s(x)$. Also rough open balls are not rough sets.

Example 4.1

Consider the rough metric space given in Example 3.1. Clearly $B_1(x) = X$, $\forall x \in X$, and

$B_r(x) = X, \quad \forall r \geq 1$. If $r < 1$, then $B_r(x) = [x]_R$.

Definition 4.2

Let $<A(X),d>$ be a rough metric space and $A \subseteq X$. A point $a \in A$ is said to be an rough interior point of A if $\exists$ a real no. $r > 0$ such that the rough open ball $B_r(a) \subset A$.

Definition 4.3

Let $<A(X),d>$ be a rough metric space and $A \subseteq X$. Then A is said to be a rough open set if every point of A is a rough interior point of A.

Clearly rough open sets are not rough sets.

Proposition 4.1

A rough open set in <A(X),d> is a union of open balls.

Proof: $\forall\, a \in A,\ \exists\, r_a > 0$ such that

$$B_{r_a}(a) \subset A.$$

$$\Rightarrow \bigcup_{a \in A} B_{r_a}(x) \subseteq A \qquad \text{------------------------} \qquad (1)$$

Again, $\forall\, a \in A,\ a \in B_{r_a}(a)$.

$$\Rightarrow A \subseteq \bigcup_{a \in A} B_{r_a}(a) \qquad \text{------------------------} \qquad (2)$$

From (1) and (2) result follows

Proposition 4.2

Let <A(X),d> be a rough metric space.

Then (i) ϕ is rough open set

(ii) X is rough open set.

Proof : Straightforward.

Proposition 4.3

If <A(X),d> be a rough metric space, then

(i) the union of an arbitrary collection of rough open sets of X is a rough open set of X

(ii) the intersection of a finite collection of rough open sets of X is a rough open set set of X.

Proof(i): Let $\{A_i : i \in I\}$ be an arbitrary collection of rough open sets of X.

Let $B = \bigcup_{i \in I} A_i$, and $a \in B$ be an arbitrary point.

$\Rightarrow a \in A_i$ for at least one $i \in I$

$\Rightarrow \exists\ r > 0$ such that $B_r(a) \subset A_i$

$\Rightarrow B_r(a) \subset B$.

=> a is a rough interior point of B.

=> B is rough open.

Proof(ii): Let $A_1, A_2, \ldots, A_n$ be a finite collection of rough open sets of X. Let $B = \bigcap_{i=1}^{n} A_i$ and $a \in B$.

=> $a \in A_i \quad \forall\, i = 1,2,\ldots,n.$

=> $\forall i, \quad \exists\, r_i > 0$ s.t. $B_{r_i}(a) \subset A_i$

Suppose, $r = \min r_i$

=> $B_r(a) \subset A_i \quad \forall\, i.$

=> $B_r(a) \subset B$

=> a is a rough interior point of B.

=> B is rough open.

Corollary 4.1

We can see by an example that the intersection of an arbitrary collection of rough open sets need not be rough open.

For this, consider X = set of real numbers and R is the equivalence relation defined on X such that $\forall x,y \in X$, xRy if $x = y$. Choose the rough metric $d = |x-y|$. Clearly, <A(X),d> is a rough metric space. Now consider the sequence $\{G_n\}$ of rough open sets

$G_n = (-\frac{1}{n}, \frac{1}{n})$, $n \in N$ (set of natural numbers).

Clearly, $\bigcap_{n=1}^{\infty} G_n$ is not rough open.

Proposition 4.4

A rough open ball in <A(X),d> is a rough open set.

Proof : Straightforward.

REFERENCES

[1] D.Dubois and H.Prade. Rough fuzzy sets and fuzzy rough sets, Int.Jour.Gen.Systems. 17(1989)191-209.

[2] S. Nanda. and S. Majumdar, Fuzzy rough sets, Fuzzy Sets and Systems. 45 (1992) 157-160.

[3] T. B. Iwinski, Algebraic approach to rough sets. Bull. Pol. Aca. Sci. -Mathematics 35, 9-10 (1987), 673-683.

[4] Z. Pawlak, Rough sets, Basic notions. ICS PAS Reports, 436 (1981).

[5] Z. Pawlak, Rough sets, Power set hierarchy, ICS PAS Reports, 470 (1982).

[6] Z. Pawlak, Rough sets, Algebraic and topological approach. ICS PAS Reports, 482 (1982).

[7] Z. Pawlak, Some remarks on Rough Sets, Bull. Pol. Ac.: Tech 33 (1985).

[8] Z. Pawlak, Rough Sets, Inter.J. Inf. Comp.Sc., 11(5) (1982) 341-356.

[9] Z. Pawlak, Rough sets and fuzzy sets, Fuzzy Sets and Systems. 17(1985) 99 - 102.