FUZZY HYBRID FIXED POINT THEOREMS ON MENGER PROBABILISTIC METRIC SPACES

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ABSTRACT: This paper bring forward the concept of fuzzy hybrid fixed point and concept of common fuzzy hybrid fixed point on Menger PM-Spaces, and studied common fuzzy hybrid fixed point theorems for a sequence of fuzzy mappings, The results presented improve and generalige the corresponding results of O. Hadzic, J. Z. FAN, S. S. CHANG, etc.

KEY WORDS: Menger probabilistic metric Space, fuzzy mapping, Sequence of fuzzy mapping, fuzzy hybrid fixed point, Common fuzzy hybrid fixed point.
1991 **AMS SYBJECT CLASSIFICATION CODES**: 54H25, 47H10.

1 PRELIMINARIES

Throughout this paper, we assume that (E,F,Δ) is a Menger probabilistic metric Space (Shortly a M-PM-Space) With the (ε,λ)—topology τ[1], let $R=(-\infty,+\infty)$, Z^+ is the family of all positive integers, $CB(E)$ is the family of all nonempty τ-closed Subsets of E, $C(E)$ is the family of all nonempty τ-Compact subsets of E.

DEFINITION 1.1 A t-norm Δ is said to be of h-type if the family $\{\Delta^m(t)\}_{m=1}^{+\infty}$ of the functions $\Delta^m(t,\Delta_{m-1}(t))$, $m=1,2,\cdots,\Delta^0(t)=1$, $t\in[0,1]$, is equicontinuous at 1.

DEFINITION 1.2 Let (E,F,Δ) be a M-PM-Space, $A,B\in CB(E)$, $x\in E$, We define $F_{x,A}(t)$ and $F_{A,B}(t)$ by

$$F_{x,A}(t)=\sup_{y\in A}F_{x,y}(t),\ \forall\, t\in R, \text{and}$$

$$F_{A,B}(t)=\sup_{s<t}\Delta(\inf_{x\in A}\sup_{y\in B}F_{x,y}(s),\inf_{y\in B}\sup_{x\in A}F_{x,y}(s)),\ \forall\, t\in R$$

DEFINITION 1.3 If a function $\Phi:[0,+\infty)\to[0,+\infty)$ Satisfies: (1) it is a Strictly increasing, left-Continuity, and $\Phi(0)=0$, $\lim_{t\to+\infty}\Phi(t)=+\infty$, (2) $\sum_{n=0}^{+\infty}\Phi^n(t)<+\infty, \forall t>0$, $\Phi^n(t)$ denotes the n-the iterative function of $\Phi(t)$, We say that Φ Satisfies the condition(Φ)

DEFINITION 1.4 Let $\Phi:[0,+\infty)\to[0,+\infty)$ Satisfies the condition (1) of definition 1.3, define a function $\Psi:[0,+\infty)\to[0,+\infty)$,

$$\text{by}\quad \Psi(t)=\begin{cases}0, & t=0\\ \inf\{s>0:\Phi(s)>t\}, & t>0\end{cases}\tag{1.1}$$

It is easy to prove that $\Psi:[0,+\infty)$ is a continuous and nondecreasing function [6]

DEFINITION 1.5 a mapping $A:E\to[0,1]$ is called a fuzzy Subset Over E, We denote by W(E) the family of all fuzzy subsets Over E, a mapping $T:E\to W(E)$ is called a fuzzy mapping Over E. let $T:E\to W(E)$. $S:E\to E$, if $P\in E$ such that $Tp(P)=\max_{u\in E} Tp(u)$ and $Sp=p$, then p is called a fuzzy hybrid fixed point of T and S. Let $T_K:E\to W(E)$ $(K=1.2\cdots)$, $S=E\to E$, if $P\in E$ such that $(\bigcap_{K=1}^{+\infty}T_Kp)(p)=\max_{u\in E}(\bigcap_{K=1}^{+\infty}T_K)(u)$ and $Sp=p$, then p is called a Comnion fuzzy hybrid fixed point of $\{T_K\}$ and S.

LEMMA 1.6 [6]Let (E,F,Δ) is a M-PM-Space, Δ be a left-continuous t-norm, $A\in CB(E)$, $x,y\in E$, then: (1) for any $B\in CB(E)$ and $x\in A$, $\inf_{x\in A}\sup_{y\in B}F_{x,y}(t)\leqslant F_{x,B}(t), \forall t\in R$; (2)$F_{x,A}(t)=1$, for $\forall t>0$, if and only if $x\in A$; (3) $F_{x,A}(t_1+t_2)\geqslant\Delta(F_{x,y}(t_1),F_{y,A}(t_2))$, for $\forall\ t_1,t_2>0$; (4)$F_{x,A}(t)$ is a left-continuous functions at t.

LEMMA 1.7 [6]Let $\Phi:[0,+\infty)\to[0,+\infty)$ satisfy the condition (Φ) and let Ψ be defind by (1.1), then: (1) $\Phi(t)<t, \forall t>0$; (2)$\Phi(\Psi(t))\leqslant t$ and $\Psi(\Phi(t))=t\ \forall t\geqslant0$; (3) $\Psi(t)\geqslant t, \forall t\geqslant 0$; (4) $\lim_{n\to+\infty}\Psi^n(t)=+\infty, \forall t>0$.

LEMMA 1.8 [6]Let (E,F,Δ) is a M-PM-Space, Where Δ is a t-norm of h-type, if a sequence $\{x_K\}$ in E Satisfies the following Condition: for $\forall K\in Z^+$ and $t>0$

$$F_{x_K,x_{K+1}}(t)\geqslant F_{x_0,x_1}(\Psi^K(t))\tag{1.2}$$

where Φ is a function satisfying the condition (Φ) and Ψ os defined by (1.1), then the Sequence $\{x_K\}$ is a τ-Canchy sequance in E.

2 MAIN RESULTS

Let $T_K: E \to W(E)$, $O_K(x): E \to (0,1]$ $(K=1,2,\cdots)$, throughout this paper, We denote $(T_K x)O_K(x)$ by $\widetilde{T}_K x$ Set, i.e. $\widetilde{T}_K x = (T_K x)O_K(x) = \{u \mid T_K x(u) \geqslant O_K(x)\}$, $\forall\, x \in E$, $(K=1,2,\cdots)$。

LEMMA 2.1 Let (E,F,Δ) be a τ-complete M-PM-Space, $S: E \to E$ be a singte-vatuld continuous mapping such that $\forall\, x,y \in E$

$$F_{x,Sy}(t) \leqslant F_{Sx,Sy}(t), \forall\, t > 0 \qquad (2.1)$$

Then $S(E) = \{Z \mid Z = Sx, x \in E\} = F(E) = \{x \mid x = Sx, x \in E\}$

PROOF It is Obvious that $F(E) \subseteq S(E)$, Next, We prove that $S(E) \subseteq F(S)$, $\forall\ Z_1 \in S(E)$, $\exists\ x_1 \in E$ with $Z_1 = Sx_1$, by (2.1), $F_{Sz_1,Sx_1}(t) \geqslant F_{z_1,Sx_1}(t) = H(t)$, $\forall\, t \in R$, therefore $Sz_1 = Sx_1 = Z_1$, $Z_1 \in F(E)$, thus $S(E) \subseteq F(S)$, by $F(E) \subseteq S(E)$ and $S(E) \subseteq F(E)$, We have $F(E) = S(E)$.

THEOREM 2.1 Let (E,F,Δ) be a τ-complete M-PM-Space, where Δ is a left-continuous t-norm of h-type, let $S: E \to E$ be a Single-Vatued continuous mapping such that (2.1), $T_K: E \to W(E)$ $(K=1,2,\cdots)$ be a Sequence of fuzzy mappings,

(1) If there exists a Sequence of funetions $O_K(x): E \to (0,1]$ $(K=1,2,\cdots)$ Such that for any $x \in E$, $K \in Z^+$, $\widetilde{T}_K x \in CB(E)$, $S(\widetilde{T}_K x) = \widetilde{T}_K Sx$ and for any $K, L \in Z^+$, $x,y \in E$, $u \in \widetilde{T}_K x$ there exists $v \in \widetilde{T}_L y$ such that

$$F_{u,v}(\Phi(t)) \geqslant \min\{F_{x,y}(t), F_{x,\widetilde{T}_K x}(t), F_{y,\widetilde{T}_L y}(t),\}, \forall\, t > 0 \qquad (2.2)$$

Where Φ satisfies the condition (Φ), then there exists $P \in E$ Such that $p = Sp$ and $(\bigcap_{K=1}^{+\infty} T_K p)(P) \geqslant \min_{K \geqslant 1}\{O_K(p)\}$.

(2) when $O_K(x) = \max_{u \in E} T_K x(u)$ $(K=1,2,\cdots)$ be a sequence of functions satisfying the Condition (1), then $\{T_K\}$ and S have a Common fuzzy hybrid fixed Point P in E.

PROOF By $S: E \to E$ is continuous and lemma 2.1, $S(E) = F(S)$ is τ-closed subset of E, therefore $(S(E), F, \Delta)$ be a τ-complete M-PM-space, we prove that $\forall\, x \in S(E)$, $\widetilde{T}_K x \subseteq S(E)$ $(K=1,2,\cdots)$, in fact, for any $x \in S(E)$, by $x = Sx$, $S(\widetilde{T}_K x) = \widetilde{T}_K Sx$, We have $\widetilde{T}_K x = \widetilde{T}_K Sx = S(\widetilde{T}_K x) \subseteq S(E)$. Take $x_0 \in S(E)$ and $x_1 \in \widetilde{T}_1 x_0 \subseteq S(E)$, by definition 1.2, lemma 1.7, (2.2), $\exists\ x_2 \in \widetilde{T}_2 x_1$ such that

$$\begin{aligned} F_{x_1,x_2}(t) &\geqslant F_{x_1,x_2}(\Phi(\Psi(t))) \\ &\geqslant \min\{F_{x_0,x_1}(\Psi(t)), F_{x_0,\widetilde{T}_1 x_0}(\Psi(t)), F_{x_1,\widetilde{T}_2 x_1}(\Psi(t))\} \\ &\geqslant \min\{F_{x_0,x_1}(\Psi(t)), F_{x_1,x_2}(\Psi(t))\} \forall\, t > 0 \qquad (2.3) \end{aligned}$$

Where $\Psi(t)$ is defined by (1.1), Using (2.3) repeatedly, We have

$$F_{x_1,x_2}(t) \geqslant \min\{F_{x_0,x_1}(\Psi(t)), F_{x_0,x_1}(\Psi^2(t)), F_{x_1,x_2}(\Psi^2(t))\}$$

$\geqslant \min\{Fx_0,x_1(\Psi(t)),Fx_1,x_2(\Psi^2(t))\}\geqslant\cdots\geqslant$

$\geqslant \min\{Fx_0,x_1(\Psi(t)),Fx_1,x_2(\Psi^K(t))\}$

by lemma 1.7, $Fx_1,x_2(\Psi^K(t))\to 1$ $(K\to+\infty)$, We have $Fx_1,x_2(t)\geqslant Fx_0,x_1(\Psi(t)),\forall\ t>0$

Taking this procedure repeatedly, We can define a sequence $\{x_K\}$ in E Satisfying

$$x_{K+1}\in \widetilde{T}_{K+1}x_K \textit{ and } Fx_K,x_{K+1}(t)\geqslant Fx_{K-1},x_K(\Psi(t)),\forall\ t>0 \quad (2.4)$$

Thus, for any $K\in Z^+$ and $t>0$, We have

$$Fx_K,x_{K+1}(t)\geqslant Fx_{K-1},x_K(\Psi(t))\geqslant\cdots\geqslant Fx_0,x_1,(\Psi^K(t)) \quad (2.5)$$

by lemma 1.8 $\{x_K\}$is a Cauchy Sequance in S(E), since $(S(E),F,\Delta)$is τ-complete, therefore $\exists$ $P\in S(E), P=\lim\limits_{K\to\infty}x_K$。

Next, We prove that $P\in\widetilde{T}_{L}p(L=1,2,\cdots)$ $\forall\ t>0, L\in Z^+, \varepsilon\in(o,t)$, form (2.4), (2.5) and (2.2), We have

$$Fx_{K+1},\widetilde{T}_{LP}(t-\varepsilon)\geqslant Fx_{K+1},\widetilde{T}_{LP}(\Phi(\Psi(t-\varepsilon)))$$

$$=\sup_{y\in\widetilde{T}_{LP}}Fx_{K+1},y(\Phi(\Psi(t-\varepsilon)))$$

$$\geqslant\min\{Fx_K,p(\Psi(t-\varepsilon)),Fx_K,\widetilde{T}_{K+1}x_K(\Psi(t-\varepsilon)),Fp,\widetilde{T}_{LP}(\Psi(t-\varepsilon))\}$$

$$\geqslant\min\{Fx_K,p(\Psi(t-\varepsilon)),Fx_K,x_{K+1}(\Psi(t-\varepsilon)),Fp,\widetilde{T}_{LP}(\Psi(t-\varepsilon))\}$$

$$\geqslant\min\{Fx_K,p(\Psi(t-\varepsilon)),Fx_0,x_1(\Psi^{K+1}(t-\varepsilon)),Fp,\widetilde{T}_{LP}(\Psi(t-\varepsilon))\}$$

IF $Fp\widetilde{T}_{LP}(\Psi(t-\varepsilon))=1$, by $K\to\infty$, $Fx_K,p(\Psi(t-\varepsilon))\to1$, $Fx_0,x_1(\psi^{K+1}(t-\varepsilon))\to 1$, We have $Fp,\widetilde{T}_{LP}(t-\varepsilon)\geqslant1$, by the arbitrines of $\varepsilon\in(0,1)$, We have $Fp,\widetilde{T}_{LP}(t)=1$ $(L=1,2,\cdots),\forall\ t>0$, thus $P\in\widetilde{T}_{LP}(L=1,2,\cdots)$

If $Fp,\widetilde{T}_{LP}(t)\geqslant Fp,\widetilde{T}_{LP}(\Psi(t))\geqslant\cdots\geqslant Fp\widetilde{T}_{LP}(\Psi^n(t))$, therefore as$\to+\infty$, $Fp\widetilde{T}_{LP}(t)=1,\forall\ t>0$, thus $P\in\widetilde{T}_{LP}=(T_{LP})O_L(p)$ $(L=1,2,\cdots)$, $T_{LP}(p)\geqslant O_L(p)\geqslant\min\limits_{L\geqslant1}\{O_L(p)\}(L=1,2,\cdots)$, thus $(\bigcap\limits_{L=1}^{+\infty}T_{LP})(p)=\min\limits_{L\geqslant1}T_{LP}(p)\geqslant\min\limits_{L\geqslant1}\{O_L(p)\}$.

When $O_L(x)=\max\limits_{u\in E}T_Lx(u)$, $(\bigcap\limits_{L=1}^{+\infty}T_{LP})(p)\geqslant\min\limits_{L\geqslant1}\{O_L(p)\}=\min\limits_{L\geqslant1}\ \max\limits_{u\in E}T_{LP}(u)\geqslant\min\limits_{L\geqslant1}T_{LP}(u)=(\bigcap\limits_{L=1}^{+\infty}T_{LP})(u),\forall\ u\in E$. Thus $(\bigcap\limits_{L=1}^{+\infty}T_{LP})(p)\geqslant\max\limits_{u\in E}(\bigcap\limits_{L=1}^{+\infty}T_{LP})(u)\geqslant(\bigcap\limits_{L=1}^{+\infty}T_{LP})(p)$

$\therefore(\bigcap\limits_{L=1}^{+\infty}T_{LP})(p)=\max\limits_{u\in E}(\bigcap\limits_{L=1}^{+\infty}T_{LP})(u)$, moreover $P\in S(E)=F(S), P=Sp$, Thus P is a common fuzzy hybrid fixed point of $\{T_K\}$ and S

Taking $\Phi(t)=kt(0<K<1)$ in Theorem 2.1, We have

THEOREM 2.2 Let$(E,F,\Delta),\Delta,S:E\to E,T_K:E\to W(E),O_K(x):E\to(0,1]$ $(K=1,2,\cdots)$ satisfies the conditions of theoren 2.1, moreover for ang k, L

$\in Z^+$, $x,y\in E$, $u\in \widetilde{T}_K x$ $\exists$ $u\in \widetilde{T}_L y$ such that

$$Fu,v(Kt) \geqslant \min\{Fx,y(t),Fx,\widetilde{T}_K x(t),Fy,\widetilde{T}_L y(t)\} \forall t>0 \quad (2.6)$$

$K\in(0,1)$ is a constant. Then the conclusion of theorem 2.1 remains true.

Let $G_K: E\to CB(E)(K=1,2,\cdots)$, $\forall x\in E$, take $T_K x(u)=\begin{cases}1, u\in G_K x\\ 0, u\in G_K x\end{cases}$, $\forall u\in E$, $O_K(x)\equiv 1$, by theorem 2.2, We have

COROLLARY 2.1 Let (E,F,Δ), Δ, $S: E\to E$ Satisfies the conditions of theorem 2.1, Let $G_K: E\to CB(E)$ $(K=1,2,\cdots)$ Satisfies $\forall x\in E$, $G_K x\in CB(E)$ and for any $K.L\in Z^+$ $x,y\in E$, $u\in G_K x$, $\exists$ $v\in G_L y$ such that

$$Fu,v(kt) \geqslant \min\{Fx,y(t),Fx,G_K x(t),Fy,G_L y(t)\} \forall t>0 \quad (2.7)$$

$K\in(0,1)$ is a constant

Then $\exists$ $P\in E$, $p=Sp$ and $P\in \bigcap_{K=1}^{+\infty} G_K p$

REMARK 2.1 Since the set-valued mapping is the special cases of fuzzy mapping, moreover $S=I$ be the identity operator on E, S satisfies conditions (2.1). therefore the main results of [3,4,5,6] are all the special cases of theorem 2.1. The Corollary 2.1 improve and generalige fixed point theoresn of O. HadŽic。

REFERENCES

[1] B. Schweizer and A, Sklar, probabilistic metric spaces, North-Holand, 1983.

[2] O。HadŽic, Fixed point theorems for multi-valued mapping in probabilistic metric space, Mat. Vesnik, 3(16)(31)(1979)125-133.

[3] O, HadŽic, Fixed point theorems for multi-valued mappings in Some classes of fuzzy metric spaces, Fuzzy Sets and systems, 29(1989)115-125。

[4] Jin-Xuan Fang, A note on fixed point theorems of HadŽic, Fuzzy Sets and systems 48(1992) 391-395.

[5] Jin-Xuan Fang, An improvement of HadŽic's fixed point theorem, Journal of Mathematical Research and Exposition, Vol. 14 No. 4, Nov, 1994, 565-568.

[6] S. S. CHANG, Y. J. CHO, S. M. KANG AND J. X. FAN, common fixed point theorems for multi-uatued mappings in menger PM-Spaces, Math. Japonica, 40, No. 2(1994), 289-293。