COMMON FIXED DEGREE FOR A SEQUENCE OF g-NONEXPANSIVE TYPE FUZZY MAPPINGS

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ABSTRACT: In this paper the common fixed degree for a sequence of g-nonexpansive type fuzzy mappings are studied. The results presented improve and generalize the corresponding recent important results.

KEY WORDS AND PHRASES: Fuzzy analysis, fuzzy mapping, nonexpansive type fuzzy mapping, fixed degree, common fixed degree.

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1 PRELIMINARIES

Throughout this paper, let $(E, \|\cdot\|)$ be a Banach space, $D \subseteq E$, $C(E)$ be collection of all non-empty compact subsets of E, $R = (-\infty, +\infty)$, Z^+ be the set of all positive integers, a mapping $A: D \to [0,1]$ is called a fuzzy subset over D, we denote by $\mathscr{F}(D)$ the family of all fuzzy subsets over D, a mapping $F: D \to \mathscr{F}(D)$ is called fuzzy mapping over D, let $A \in \mathscr{F}(D)$ $\alpha \in [0, 1]$, set $A_\alpha = \{x \mid A(x) \geqslant \alpha, x \in D\}$ is called the α-cut set of A.

DEFINITION 1. 1. Let $\{F_K: D \to \mathscr{F}(D)\}_{K=1}^{+\infty}$ be a sequence of fuzzy mappings, if there exsits a sequence of functions $\{O_K(x): D \to (0,1]\}_{K=1}^{+\infty}$ such that for all $x \in D$, $(F_K x)_{O_K(x)} \in C(E)$ $(K = 1, 2, \cdots)$, then we say that $\{F_K\}_{K=1}^{+\infty}$ for $\{O_K(x)\}_{K=1}^{+\infty}$ satisfies the condition (A).

Throughout this paper we denote $(F_K x)_{O_K(x)}$ by $\widetilde{F}_K x$ set, i. e., $\widetilde{F}_K x = (F_K x)_{O_K(x)}$ for all $x \in D$, all $K \in Z^+$.

DEFINITION 1. 2. Let $g: D \to D$ be a single-valued mapping, $\{F_K: D \to \mathscr{F}(D)\}_{K=1}^{+\infty}$ be a sequence of fuzzy mappings, if there exsits a sequence of functions $\{O_K(x): D \to (0, 1]\}_{K=1}^{+\infty}$ such that $\{F_K\}_{K=1}^{+\infty}$ satisfies the condition

(B) : for any $K, L \in Z^+, x, y \in D, Ux \in \widetilde{F}_K x$ there exsits $Vy \in \widetilde{F}_L y$ such that

$$\| Ux - Vy \| \leqslant r \| g(x) - g(y) \| , r \in (0,1) \qquad (1.1)$$

Then we say that $\{F_K\}_{K=1}^{+\infty}$ for $\{O_K(x)\}_{K=1}^{+\infty}$ be the sequence of g-contractive type fuzzy mapping. If the sequence of set-valued mappings $\{T_K : D \to 2^D\}_{K=1}^{+\infty}$ satisfies the condition (B), then we say tha $\{T_K\}_{K=1}^{+\infty}$ be a sequence of g-contractive type set-valued mappings.

DEFINITION 1. 3. When $r = 1$ in (1. 1) of the Definition 1. 2, we say that $\{F_K\}_{K=1}^{+\infty}$ for $\{O_K(x)\}_{K=1}^{+\infty}$ be a sequence of g-nonexpansive type fuzzy mappings.

DEFINITION 1. 4. Let $\{F_K : D \to \mathscr{F}(\mathrm{D})\}_{K=1}^{+\infty}$ be a sequence of fuzzy mappings, $p \in D$, $(\bigcap_{K=1}^{+\infty} F_K p)(p)$ is called the common fixed degree of p for $\{F_K\}_{K=1}^{+\infty}$. In particular, if $(\bigcap_{K=1}^{+\infty} F_K p)(p) = \max_{u \in D} (\bigcap_{K=1}^{+\infty} F_K p)(u)$, then we say that $\{F_K\}_{K=1}^{+\infty}$ has the maximun common fixed degree at p, or that p is a common fixed point of $\{F_K\}_{K=1}^{+\infty}$.

2 MAIN RESULTS

THEOREM 2. 1. Let $(E, \| \cdot \|)$ be a Banach Space, $D \subseteq E$ be a nonempty weakly closed subset of E, $g : D \to D$ be a nonexpansive mapping, $\{F_K : D \to \mathscr{F}(\mathrm{D})\}_{K=1}^{+\infty}$ be a sequence of fuzzy mappings. If there exsits a sequence of functions $\{O_K(x) : D \to (0,1]\}_{K=1}^{+\infty}$ such that $\{F_K\}_{K=1}^{+\infty}$ satisfies the condition (Δ_1) : $\{F_K\}_{K=1}^{+\infty}$ for $\{O_K(x)\}_{K=1}^{+\infty}$ satisfies the condition (A) and be a sequence of g-contractive type fuzzy mappings, then there exsits $p \in D$ such that the common fixed degree of p for $\{F_K\}_{K=1}^{+\infty} \geqslant \min_{i \geqslant 1}\{O_K(x)\}$. In particular, if $\{F_K\}_{K=1}^{+\infty}$ for $\{O_K(x) = \max_{u \in D} F_K x(u)\}_{K=1}^{+\infty}$ satisfies the condition (Δ_1), then there exsists $p \in D$, p be a common fixed point of $\{F_K\}_{K=1}^{+\infty}$.

PROOF. By assumption, $\{F_K\}_{K=1}^{+\infty}$ for $\{O_K(x)\}_{K=1}^{+\infty}$ satisfies the condition (A) and be a sequence of g-contractive type fuzzy mappings, $g : D \to D$ be a nonexpansive mapping, for any $x_0 \in D, \widetilde{F}_1 x_0 \in C(E)$, for any $x_1 \in \widetilde{F}_1 x_0$, there exsits $x_2 \in \widetilde{F}_2 x_1$ such that:

$$\| x_2 - x_1 \| \leqslant r \| g(x_1) - g(x_0) \| \leqslant r \| x_1 - x_0 \| .$$

for $x_2 \in \widetilde{F}_2 x_1$ there exsits $x_3 \in \widetilde{F}_3 x_2$ such that

$$\| x_3 - x_2 \| \leqslant r \| g(x_2) - g(x_1) \| \leqslant r \| x_2 - x_1 \| .$$

Taking this procedure repeatedly, we can obtain a sequence $\{x_N\} \subseteq D$ such that $x_{N+1} \in \widetilde{F}_{N+1} x_N$ moreover

$$\| x_{N+1} - x_N \| \leqslant r \| g(x_N) - g(x_{N-1}) \| \leqslant r \| x_N - x_{N-1} \| .$$

$$\| x_{N+1} - x_N \| \leqslant r \| x_N - x_{N-1} \| \leqslant \cdots\cdots \leqslant r^N \| x_1 - x_0 \| , r \in (0,1).$$

$$\| x_{N+M} - x_N \| \leqslant \sum_{K=1}^{M} \| x_{N+K} - x_{N+K-1} \| \leqslant \sum_{K=1}^{M} r^{N+K-1} \| x_1 - x_0 \|$$

$$\leqslant \frac{r^N}{1-r} \| x_1 - x_0 \| \qquad \forall\, M \in Z^+$$

it is easy to see that $\{x_N\}$ is a cauchy sequence in D. By $(E, \| \cdot \|)$ is a Banach space, there exsits $p \in E$ such that $p = \lim\limits_{N \to \infty} x_N$, by D is a nonempty weakly closed subset of E, $\lim\limits_{N \to +\infty} x_N = p$ implies $w\text{-}\lim\limits_{N \to \infty} x_N = p$, therefore $p \in D$.

Next, we prove that $p \in \widetilde{F}_M p$ $(M = 1,2,\cdots)$, for any $M \in Z^+$, by $x_N \in \widetilde{F}_N x_{N-1}$, there exsits $v_N \in \widetilde{F}_M p$ such that:

$$\| x_N - v_N \| \leqslant r \| g(x_{N-1}) - g(p) \| \leqslant r \| x_{N-1} - p \|$$

therefore $\| x_N - v_N \| \to 0$ $(N \to +\infty)$, thus

$$\| v_N - p \| \leqslant \| v_N - x_N \| + \| x_N - p \| \to 0 \quad (N \to +\infty)$$

i. e. $\lim\limits_{N \to \infty} v_N = p$, moreover $v_N \in \widetilde{F}_M p \in C(E)$, we have $p \in \widetilde{F}_M p = (F_M p)_{O_M(p)}$

By $p \in (F_M p)_{O_M(p)}$ we have $F_M p(p) \geqslant O_M(p) \geqslant \min\limits_{M \geqslant 1}\{O_M(p)\}$ $(M = 1,2,\cdots)$, therefore $(\bigcap\limits_{K=1}^{+\infty} F_K p)(p) = \min\limits_{K \geqslant 1} F_K p(p) \geqslant \min\limits_{K \geqslant 1}\{O_K(p)\}$, i. e. the common fixed degree of p for $\{F_K\}_{K=1}^{+\infty} \geqslant \min\limits_{K \geqslant 1}\{O_K(p)\}$. In particular, if $\{F_K\}_{K=1}^{+\infty}$ for $\{O_K(x) = \max\limits_{u \in D} F_K x(u)\}_{K=1}^{+\infty}$ satisfies the condition (Δ_1), we have $(\bigcap\limits_{K=1}^{+\infty} F_K p)(p) \geqslant \min\limits_{K \geqslant 1}\{O_K(p)\} = \min\limits_{K \geqslant 1}\{\max\limits_{u \in D} F_K p(u)\} \geqslant \min\limits_{K \geqslant 1} F_K p(u) = (\bigcap\limits_{K=1}^{+\infty} F_K p)(u)$ (for any $u \in D$), thus $(\bigcap\limits_{K=1}^{+\infty} F_K p)(p) \geqslant \max\limits_{u \in D}(\bigcap\limits_{K=1}^{+\infty} F_K p)(u) \geqslant (\bigcap\limits_{K=1}^{+\infty} F_K p)(p)$, therefore $(\bigcap\limits_{K=1}^{+\infty} F_K p)(p) = \max\limits_{u \in D}(\bigcap\limits_{K=1}^{+\infty} F_K p)(u)$, i. e. p be a common fixed point of $\{F_K\}_{K=1}^{+\infty}$.

COROLLARY 2. 2. Let $(E, \| \cdot \|)$, $D, g: D \to D$ satisfy the conditions of Theorem 2. 1, let $\{T_K : D \to 2^D\}_{K=1}^{+\infty}$ be a sequence of set-valued mappings. If $\{T_K : D \to 2^D\}_{K=1}^{+\infty}$ be a sequence of g-contractive type set-valued mappings, moreover for any $x \in D, T_K x \in C(E)$ $(K = 1, 2, \cdots)$, then there exsits $p \in D, p \in \bigcap_{K=1}^{+\infty} T_K p$, i. e. p be a common fixed point of $\{T_K\}_{K=1}^{+\infty}$.

THEOREM 2. 3. Let $(E, \| \cdot \|)$ be a Banach space whick satisfies opial's condition ([3], [4]), D be a nonempty weakly closed star-shaped subset of E, M be a weakly compact subset of E, $g : D \to D$ be a nonexpansive mapping, $\{F_K : D \to \mathscr{F}(\mathrm{D})\}_{K=1}^{+\infty}$ be a sequence of fuzzy mappings. If there exists a sequence of functions $\{O_K(x) : D \to (0, 1]\}_{K=1}^{+\infty}$ such that $\{F_K\}_{K=1}^{+\infty}$ satisfies the condition (Δ_2): $\{F_K\}_{K=1}^{+\infty}$ for $\{O_K(x)\}_{K=1}^{+\infty}$ satisfies the condition (A) and be a sequence of g-nonexpansive type fuzzy mappings, for any $x \in D, \widetilde{F}_1 x \subseteq M$.

Then there exists $p \in D$, the common fixed degree of p for $\{F_K\}_{K=1}^{+\infty} \geqslant \min_{K \geqslant 1}\{O_K(p)\}$. In particular, if $\{F_K\}_{K=1}^{+\infty}$ for $\{O_K(x) = \max_{u \in D} F_K x(u)\}_{K=1}^{+\infty}$ satisfies the condition (Δ_2), then $\{F_K\}_{K=1}^{+\infty}$ has a common fixed point P in D.

PROOF. Let $t_N \in (0, 1)$ $(N = 1, 2, \cdots)$, $\lim_{N \to \infty} t_N = 1$, x_0 be the star-centre of D, for each $N \geqslant 1$, define the set-valued mapping $T_K^N : D \to 2^D (K = 1, 2, \cdots)$ by setting

$$T_K^N x = t_N \widetilde{F}_K x + (1 - t_N) x_0 = \{t_N u + (1 - t_N) x_0 \mid u \in \widetilde{F}_K x\}, x \in D$$

for any $x \in D$, by $\widetilde{F}_K x \in C(E)$, therefore $T_K^N x \in C(E)$, for any $K, L \in Z^+$, any $x, y \in D$, any $U_X^N \in T_K^N x$, there exists $Ux \in \widetilde{F}_K x$ such that $U^N x = t_N Ux + (1 - t_N) x_0$, by $\{F_K\}_{K=1}^{+\infty}$ be a sequence of g-nonexpansive type fuzzy mappings, therefore there exists $Vy \in \widetilde{F}_L y$ such that $\| Ux - Vy \| \leqslant \| g(x) - g(y) \|$, thus $V^N y = t_N Vy + (1 - t_N) x_0 \in T_L^N y$ satisfies

$$\| U^N x - V^N y \| = t_N \| Ux - Vy \| \leqslant t_N \| g(x) - g(y) \|, t_N \in (0, 1)$$

which implies that $\{T_K^N\}_{K=1}^{+\infty}$ be a sequence of g-contractive type set-valued mappings, by corollary 2. 2 there exists $p_N \in \bigcap_{K=1}^{+\infty} T_K^N p_N$, i. e. $p_N \in T_K^N p_N \subseteq D$ $(K = 1, 2, \cdots)$. Let $p_N = t_N u_N + (1 - t_N) x_0$, $u_N \in \widetilde{F}_{K p_N} \in C(E)$ $(K = 1, 2, \cdots)$, $p_N - u_N = (1 - t_N)(x_0 - U_N)$, since $\{u_N\} \subseteq \widetilde{F}_1 p_N \subseteq M$, $\{u_N\}$ is bounded, $\{x_0 - U_N\}$ is bounded, therefore $p_N - u_N \to 0$ $(N \to + \infty)$, let $g_N = p_N - u_N$, $p_N - g_N = u_N \in \widetilde{F}_1 p_N \subseteq M$, M is a weakly compact subset, therefore there exists $\{P_M - q_M\} \subseteq \{P_N - q_N\}$ such that $w\text{-}\lim_{M \to \infty} (P_M - q_M) = P \in M$, by $q_M \to 0 (M \to + \infty)$ we have $w\text{-}\lim_{M \to \infty} P_M = P \in D$.

Next we prove that for any $L \in Z^+, p \in \widetilde{F}_L p$. In fact, for any $M \geqslant 1$, by $p_M - q_M = u_M \in \widetilde{F}_K p_M$ and $\{F_K\}_{K=1}^{+\infty}$ be a sequence of g-nonexpansive type fuzzy mappings, there exists $v_M \in \widetilde{F}_L p$ such that $\| u_M - v_M \| \leqslant \| g(p_M) - g(p) \| \leqslant \| p_M - p \|$, therefore $\| p_M - (q_M + v_M) \| \leqslant \| p_M - p \|$, since $v_M \in \widetilde{F}_L p \in C(E)$, therefore there exists $\{v_S\} \subseteq \{v_M\}$ such that $\lim_{S \to \infty} v_S = v \in \widetilde{F}_L p$, thus $q_S + v_S \to v (S \to +\infty)$, we have $\liminf_S \| p_S - v \| \leqslant \liminf_S \| p_S - p \|$. Sine $w\text{-}\lim_{S \to \infty} p_S = p$, by opial's condition, we have $p = v \in \widetilde{F}_L p$ (for any $j \in Z^+$), therefore $p \in \bigcap_{L=1}^{+\infty} \widetilde{F}_L p$. By $p \in \widetilde{F}_L p$ $(L = 1, 2, \cdots)$, the same as proof of 2. 1, we obtain the conclusion of Theorem 2. 3.

COROLLARY 2. 4. Let $(E, \| \cdot \|), D, g: D \to D, M$ satisfy the conditions of Theorem 2. 3. If $\{T_K: D \to 2^D\}_{K=1}^{+\infty}$ satisfies the condition (Δ_3) : for any $x \in D, T_K x \in C(E)$, moreover $\{T_K\}_{K=1}^{+\infty}$ be a sequence of g-nonexpansive type set-valued mappings, then $\{T_K\}_{K=1}^{+\infty}$ has a common fixed point in D.

REMARK 2. 5. The main results of [4, 5] are the special cases of Corollary 2. 4, the main results of [2, 3] are the special cases of Theorem 2. 3.

REFERECES

[1] Chang, S. S. , Fixed degree for fuzzy mappings and a generalization of KY FAN'S theorem, Fuzzy Sets and Systems, 24(1987)103—112.

[2] Lee, B. S. and Cho, S. J. , A fixed point theorem for contractive-type fuzzy mappings, Fuzzy Sets and Systems, 61(1994) 309-312.

[3] Lee, B. S. and Kim, D. S. et al, common fixed points for nonexpansive and nonexpansive type fuzzy mapping, Internat. J. Math. & Math. Sci. Vol. 18 No. 4 (1995) 813-815

[4] Latif, A. et al, Fixed point of nonexpansive type and K-multivalued maps, Internat. J. Math. & Math. Sci. Vol. 17 No. 3 (1994) 429-436.

[5] Husain, T. , Latif, A. , Fixed points of multivalued nonexpansive maps, Internt. J. Math. & Math. Sci. Vol. 14. No. 3 (1991) 421-430.