On the product of two fuzzy measures

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Abstract

In this paper, we consider the third of the open problems presented by Z. Wang in *Fuzzy Sets and Systems,* 45(1992)223-226. The problem is that under what condition the product of two fuzzy measures with null-additivity (resp. autocontinuity and resp. uniform autocontinuity) is null-additive (resp. autocontinuous and resp. uniformly autocontinuous). We obtain that the above conclusion is true if the two fuzzy measures are mutually weakly absolutely continuous.

Definition 1[1−6]. Let X be a non-empty set and $\mathcal{A}$ be a sigma-algebra of subsets of X. A set function $\mu : \mathcal{A} \to [0, +\infty)$ is called a *fuzzy measure* if
(FM1) $\mu(\emptyset) = 0$;
(FM2) $\forall A, B \in \mathcal{A}, A \subset B \Rightarrow \mu(A) \leq \mu(B)$;
(FM3) $\forall A_n, A \in \mathcal{A}, n = 1, 2, ..., A_n \uparrow A \Rightarrow \mu(A_n) \uparrow \mu(A)$;
(FM4) $\forall A_n, A \in \mathcal{A}, n = 1, 2, ..., A_n \downarrow A \Rightarrow \mu(A_n) \downarrow \mu(A)$.

If μ is a fuzzy measure defined on a measurable space $(X, \mathcal{A})$, we call $(X, \mathcal{A}, \mu)$ a *fuzzy measure space.*

Definition 2[2,4]. Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space. μ is called *null-additive,* if $\forall A, B \in \mathcal{A}$ such that $\mu(A) = 0$, there holds $\mu(A \cup B) = \mu(B)$; μ is called *autocontinuous,* if $\forall B \in \mathcal{A}$ and $\epsilon > 0$, there exists $\delta = \delta(\epsilon, B) > 0$ such that $\forall A \in \mathcal{A}$ with $\mu(A) \leq \delta$, there holds $\mu(A \cup B) \leq \mu(B) + \epsilon$; μ is called *uniformly autocontinuous,* if $\forall \epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$, such that $\forall A, B \in \mathcal{A}$ with $\mu(A) \leq \delta$, there holds $\mu(A \cup B) \leq \mu(B) + \epsilon$; μ is called to have *property(S)*, if $\forall \{A_n\} \subset \mathcal{A}, n = 1, 2, ..., \mu(A_n) \to 0$, then there exists a subsequence $\{A_{n_k}\}$ of $\{A_n\}$ such that $\mu(\limsup_{k\to\infty} A_{n_k}) = 0$.

Proposition 3[2,4]. If a fuzzy measure is uniformly autocontinuous, then it is autocontinuous; if a fuzzy measure is autocontinuous, then it is null-additive; a fuzzy measure is autocontinuous if and only if it is null-additive and has the property(S).

Wang [2] presented the following open problem: is the product of two fuzzy measures (this product must be a fuzzy measure) with null-additivity (resp. autocontinuity and resp. uniform autocontinuity) null-additive (resp. autocontinuous and resp. uniformly autocontinuous)?

In this paper, we show that this conclusion is true if the two fuzzy measures are mutually weakly absolutely continuous.

Definition 4[1,3,5,6]. Let μ_1, μ_2 be two fuzzy measures on $(X, \mathcal{A})$. (1) μ_1 is said to be *weakly absolutely continuous* with respect to μ_2 and is denoted with $\mu_1 \ll_w \mu_2$, if $\forall A \in \mathcal{A}, \mu_2(A) = 0 \Rightarrow \mu_1(A) = 0$; (2) μ_1 is said to be *strongly absolutely continuous* with respect to μ_2 and is denoted by $\mu_1 \ll_s \mu_2$, if $\forall \epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\forall A \in \mathcal{A}, \mu_2(A) \leq \delta \Rightarrow \mu_1(A) \leq \epsilon$.

Proposition 5[1]. (1) If $\mu_1 \ll_s \mu_2$, then $\mu_1 \ll_w \mu_2$; (2) If $\mu_1 \ll_w \mu_2$ and μ_2 has property (S), then $\mu_1 \ll_s \mu_2$.

In the References listed in the paper, the above two kinds of continuity of two fuzzy measures are all called absolute continuity. However, in this paper, to distinguish these two different kinds of continuities, we use the denominations of weakly and strongly absolute continuities.

The following theorem is the main results of this paper.

Theorem 6. Let μ_1, μ_2 be two fuzzy measures on $(X, \mathcal{A})$, $\mu_1 \ll_w \mu_2$ and $\mu_2 \ll_w \mu_1$. Then:
(i) if μ_1, μ_2 are null-additive, then so is $\mu_1\mu_2$;
(ii) if μ_1, μ_2 are autocontinuous, then so is $\mu_1\mu_2$;
(iii) if μ_1, μ_2 are uniformly continuous, then so is $\mu_1\mu_2$.

Proof. (i) Let $A \in \mathcal{A}$ with $\mu_1\mu_2(A) = 0$, we have $\mu_1(A) = 0$ or/and $\mu_2(A) = 0$. In addition, since $\mu_1 \ll_w \mu_2$ and $\mu_2 \ll_w \mu_1$, then $\forall A \in \mathcal{A}, \mu_2(A) = 0$ if and only if $\mu_1(A) = 0$. We have $\mu_1(A) = 0$ and $\mu_2(A) = 0$, and therefore, $\forall B \in \mathcal{A}$, from the null-additivity of μ_1, μ_2,

$$\mu_1\mu_2(A \cup B) = \mu_1(A \cup B)\mu_2(A \cup B) = \mu_1(B)\mu_2(B) = \mu_1\mu_2(B).$$

The conclusion follows.

(ii) since $\mu_1 \ll_w \mu_2, \mu_2 \ll_w \mu_1$ and μ_1, μ_2 are autocontinuous, then $\mu_1 \ll_s \mu_2$ and $\mu_2 \ll_s \mu_1$.

It is obvious that $\mu_1\mu_2$ is null-additive. From Proposition 3, we only need to prove that $\mu_1\mu_2$ has property(S).

$\forall \{A_n\} \subset \mathcal{A}, n = 1, 2, ..., \mu_1\mu_2(A_n) \to 0$, we have $\mu_1(A_n) \to 0$ and $\mu_2(A_2) \to 0$. Indeed, if one them, say, $\mu_1(A_n) \to 0$ is not true, then there exists a subsequence $\{A_{n_k}\}$ of $\{A_n\}$ such that $\mu_1(A_{n_k}) \to a > 0$. Hence, $\mu_2(A_{n_k}) \to 0$. However, from $\mu_1 \ll_s \mu_2$ and $\mu_2(A_{n_k}) \to 0$, we know that $\mu_1(A_{n_k}) \to 0$, a contradiction.

For $\{A_n\}$ mentioned above, from prqperty(S) of μ_1, we have there exists a subsequence $\{A_{n_k}\}$ of $\{A_n\}$ such that $\mu_1(\limsup_{k\to\infty} A_{n_k}) = 0$, and hence, from the property(S) of μ_2, there exists a subsequence $\{A_{n_{k_s}}\}$ of $\{A_{n_k}\}$ such that $\mu_2(\limsup_{s\to\infty} A_{n_{k_s}}) = 0$, therefore,

$$\mu_1\mu_2(\limsup_{k\to\infty} A_{n_{k_s}}) = 0.$$

The conclusion follows.

(iii) Since $\mu_1 \ll_w \mu_2, \mu_2 \ll_w \mu_1$ and μ_1, μ_2 are uniformly autocontinuous, then $\mu_1 \ll_s \mu_2$ and $\mu_2 \ll_s \mu_1$.

For every given $\epsilon > 0$, note $\epsilon_1 = \epsilon_1(\epsilon) = \min\{\frac{\epsilon}{\mu_1(X)+\mu_2(X)+1}, 1\}$, then, from the uniform autocontinuity of μ_1, μ_2, we know that there exits $\delta_1 = \delta_1(\epsilon_1) = \delta_1(\epsilon) > 0$ [1] and $\delta_2 = \delta_2(\epsilon_1) = \delta_2(\epsilon) > 0$ such that $\forall A, B \in \mathcal{A}$,

$$\mu_1(A) \leq \delta_1 \Rightarrow \mu_1(A \cup B) \leq \mu_1(B) + \epsilon_1, \tag{1}$$

and

$$\mu_2(A) \leq \delta_2 \Rightarrow \mu_2(A \cup B) \leq \mu_2(B) + \epsilon_1. \tag{2}$$

[1]Here $\delta_1(\epsilon_1) = \delta_1(\epsilon)$ does not mean that $\epsilon_1 = \epsilon$, it notes that δ_1 depends on ϵ_1 and hence depends on ϵ.

From $\mu_1 \ll_s \mu_2$, we know that there exists $\rho_2 = \rho_2(\delta_1) = \rho_2(\epsilon) > 0$ such that $\forall A \in \mathcal{A}$,

$$\mu_2(A) \leq \rho_2 \Rightarrow \mu_1(A) \leq \delta_1. \tag{3}$$

Similarly, from $\mu_2 \ll_s \mu_1$, we know that there exists $\rho_1 = \rho_1(\delta_1) = \rho_1(\epsilon) > 0$ such that $\forall A \in \mathcal{A}$,

$$\mu_1(A) \leq \rho_1 \Rightarrow \mu_2(A) \leq \delta_2. \tag{4}$$

Take

$$\delta_3 = \delta_3(\epsilon) = min(\delta_1, \delta_2, \rho_1, \rho_2) > 0$$

and

$$\delta = \delta(\epsilon) = \delta_3^2 > 0.$$

Let $A, B \in \mathcal{A}$ with $\mu_1\mu_2(A) \leq \delta$, that is to say, $\mu_1\mu_2(A) \leq \delta_3^2$. Then, at least one of the following (5) and (6) must be true.

$$\mu_1(A) \leq \delta_3 \tag{5}$$

$$\mu_2(A) \leq \delta_3. \tag{6}$$

If (5) is true, we have

$$\mu_1(A) \leq \delta_1. \tag{7}$$

In addition, (5) implies that $\mu_1(A) \leq \rho_1$, and hence, from (4),

$$\mu_2(A) \leq \delta_2. \tag{8}$$

If (6) is true, we can similarly prove that (7) and (8) are true.

Therefore, from (1), (2), (7) and (8),

$$\begin{aligned}
\mu_1\mu_2(A \cup B) &= \mu_1(A \cup B)\mu_2(A \cup B) \\
&\leq (\mu_1(B) + \epsilon_1)(\mu_2(B) + \epsilon_1) \\
&= \mu_1(B)\mu_2(B) + (\mu_1(B) + \mu_2(B) + \epsilon_1)\epsilon_1 \\
&\leq \mu_1(B)\mu_2(B) + (\mu_1(X) + \mu_2(X) + \epsilon_1)\epsilon_1 \\
&\leq \mu_1(B)\mu_2(B) + (\mu_1(X) + \mu_2(X) + 1)[\tfrac{\epsilon}{\mu_1(X)+\mu_2(X)+1}] \\
&= \mu_1(B)\mu_2(B) + \epsilon \\
&= \mu_1\mu_2(B) + \epsilon.
\end{aligned}$$

The conclusion follows.

Remark 7. We can also show that Theorem is true if we replace $\mu_1\mu_2$ by $T(\mu_1, \mu_2)$, where T is a function from $[0, +\infty) \times [0, +\infty)$ to $[0, +\infty)$ such that
(1) T is continuous;
(2) $T(a, b) > 0$ if and only if $a > 0$ and $b > 0$;
(3) if $a \leq c, b \leq d$, then $T(a, b) \leq T(c, d)$.

This paper was written when the author was in P. R. China: Department of Mathematics, Hebei Normal College, Shijiazhuang, Hebei, 050091.

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