Specificity shift in solving fuzzy relational equations

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Abstract Studied is a system of "N" fuzzy relational equations with a max - t composition

$$x(k) \square R = y(k)$$

where the input - output fuzzy sets (x(k) and y(k)) are available while the fuzzy relation (R) needs to be determined. The solution to these equations is derived through a new paradigm of specificity shift. The main objective is to modify a level of specificity of the fuzzy sets (relational constraint) so that the modified constraints allow for the use of some standard theoretical results of the theory of fuzzy relational equations that otherwise would have been found totally unjustifiable. In more detail, the specificity of the available input fuzzy sets becomes gradually increased while an opposite tendency is observed for the output fuzzy sets.

Keywords fuzzy relational equations, approximate solutions, specificity shift, max-t composition, hybrid methods, gradient - oriented methods

1. Introduction

In this study we are concerned with an important category of fuzzy relational equations with the max - t composition

$$\mathbf{x} \square R = \mathbf{y} \tag{1}$$

where "t' is assumed to be a continuous t - norm while **x**, **y** and R are viewed as fuzzy sets and a fuzzy relation defined in finite universes of discourse. The problem of analytical solutions to these equations has been pursued in the depth; refer e.g., to the monograph by Di Nola et al (1989) as helpful source of the most extensive coverage of the area; see also Di Nola and Sessa(1993). On the applied side, these equations call for approximate solutions as quite often no analytical solutions can be generated. This pursuit has been handled with the aid of various techniques. The first paper in the area of approximate (numerical) solutions to fuzzy relational equations was published in 1983 (Pedrycz). The list below concisely summarizes representatives of the existing methods:

- ❑ numeric solutions to fuzzy relational equations (gradient - based or gradient -like methods): Pedrycz(1983), Zhang et al (1994).
- ❑ fuzzy relational equations viewed as fuzzy neural networks and the ensuing use of various learning techniques specific to neurocomputing: Blanco et al. (1994), Pedrycz (1991; 1993; 1995), Pedrycz and Rocha (1993), Saito and Mizumoto (1992), Valente de Oliveira (1993),Reyes Garcia and Bandler (1994), Ghalia and Alouani (1994).
- ❑ Data preprocessing of relational constraints: clustering, logical filtering, data elimination, etc. Gottwald (1992), Gottwald and Pedrycz (1986), Hirota (1982), Pedrycz (1985), Hirota and Pedrycz (1996, to appear),
- ❑ Structural expansion of fuzzy relational equations: Pedrycz (1985)
- ❑ Probabilistic learning: Ikoma et al. (1993)
- ❑ Evolutionary computing and hybrid methods (including GA and gradient - based techniques): Pedrycz (1994)

The approach introduced here falls under the category of data preprocessing by proposing the use to the well known theoretical results to carefully preprocessed data (relational constraints). The emerging essence is what can be called a specificity shift of relational constraints being aimed at the higher solvability of the resulting fuzzy relational equations. We briefly lay down the problem, introduce basic machinery to be used in the approach and discuss the algorithm. Numerical experiments are thoroughly reported along with some thoughts on hybrid techniques used in handling fuzzy relational equations.

2. Problem formulation

The problem is stated accordingly: Given is a collection of fuzzy data (treated as vectors in two finite unit hypercubes) $(\mathbf{x}(1), \mathbf{y}(1)), (\mathbf{x}(2), \mathbf{y}(2)), ..., (\mathbf{x}(N), \mathbf{y}(N))$ where $\mathbf{x}(k) \in [0,1]^n$ and $\mathbf{y}(k) \in [0, 1]^m$. Determine a fuzzy relation R satisfying the collection of the relational constraints (fuzzy relational equations)

$$\mathbf{x}(k) \,\Box\, R = \mathbf{y}(k) \tag{2}$$

Expressing (2) in terms of the corresponding membership functions of $\mathbf{x}(k)$, $\mathbf{y}(k)$ and R we derive

$$y_j(k) = \bigvee_{i=1}^{n} [x_i(k) t\, r_{ij}] \tag{3}$$

k=1, 2, ..., N, j=1, 2, ..., m. The emerging problem can be essentially classified as an interpolation task where the fuzzy relation R needs to go through all the already specified interpolation points (fuzzy sets). Assuming that there exists a solution to (2), the theory (Di Nola et al, 1989) provides us with the solution to the fuzzy interpolation problem given in the form of the maximal fuzzy relation with the membership function equal to

$$R = \bigcap_{k=1}^{N} (\mathbf{x}(k) \rightarrow \mathbf{y}(k)) \tag{4}$$

Note that the computations involve an intersection of the individual fuzzy relations determined via a psuedocomplement (residuation) associated with the t-norm standing in the original system of equations (2), namely $a \rightarrow b = \sup \{c \in [0,1] | \; atc \le b\}$. The main advantage lies in the simplicity, theoretical soundness, well - articulated properties and compactness of this solution. The major drawback originates from the fact that the determined result holds under a rather strong preliminary assumption about the existence of any solution to (2). Now, if this assumption is evidently violated, the quality of the obtained solution could be very low. This is additionally aggravated by the fact that the derived solution is extremal (maximal) so that even a single relational constraint may contribute to the deterioration of the final aggregate result. The use of the optimization methods leads to better approximate solutions yet the entire procedure could be quite often time - consuming. Furthermore, as there could be a multiplicity of solutions, such approaches usually identify only one of them and leaving the rest of them unknown.

In the investigated setting we are interested in making some repairs to the original relational constraints thus converting the original interpolation nodes into more feasible ones, meaning that there is a higher likelihood of finding a fuzzy relation capable of doing the interpolation of the modified constraints. This is accomplished by nonlinearly affecting the data and changing their membership values. To contrast the scheme of direct computation of R as implied by the theory refer to Fig.1 where both $\varphi(\mathbf{x})$ and $\psi(\mathbf{y})$ denote the membership functions resulting from these nonlinear transformations of the original fuzzy sets forming the interpolation nodes in the initial problem.

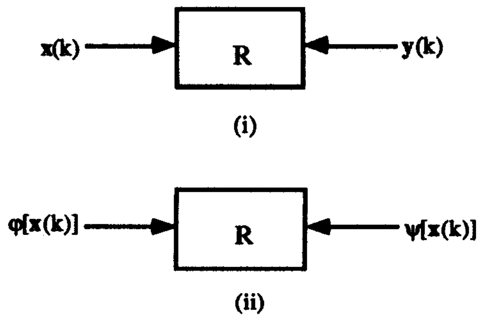

Fig.1. Determination of a fuzzy relation: (i) original data (x(k), y(k)), (ii) modified data

Thus instead of (4), one proceeds with the following expression

$$R = \bigcap_{k=1}^{N} (\varphi[x(k)] \rightarrow \psi[y(k)]) \tag{5}$$

where both $\varphi(\mathbf{x})$ and $\psi(\mathbf{y})$ are defined pointwise meaning that

$$\varphi(\mathbf{x}) = [\varphi(x_1)\ \varphi(x_2)\ \dots\ \varphi(x_n)] \tag{6}$$

and

$$\psi(\mathbf{x}) = [\psi(x_1)\ \psi(x_2)\ \dots\ \psi(x_m)] \tag{7}$$

3. The transformation functions

There are two types of the transformation functions applied to the input and output data. The first one concerning the input fuzzy sets (**x**) is defined as a continuous mapping

$$\varphi: [0, 1] \rightarrow [0, 1]$$

such that

- φ is an increasing function of its argument
- $\varphi(1) = 1$
- $\varphi(u) \leq u$

The essence of this mapping is to make the original fuzzy set more specific (when evaluated e.g., in terms of the specificity measure (Yager, 1983). Two examples of this mapping are shown in Fig.2.

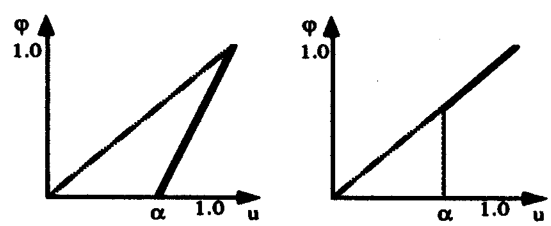

Fig.2. Two examples of specificity - increasing transformations

These functions read as

$$\varphi(u) = \begin{cases} 0, & \text{if } u < \alpha \\ \dfrac{u-\alpha}{1-\alpha}, & \text{if } u \geq \alpha \end{cases} \tag{8}$$

and

$$\varphi(u) = \begin{cases} 0, \text{ if } u < \alpha \\ u, \text{ if } u \geq \alpha \end{cases} \qquad (9)$$

For illustrative purposes, let now **x** be specified as [0.2 0.7 0.9 1.0]. Then (8) with $\alpha = 0.4$ yields

[0.0 0.5 0.83 1.0]

The same fuzzy set transformed via (9) with $\alpha = 0.4$ provides with the membership function equal to $\varphi(\mathbf{x})$ = [0.0 0.7 0.9 1.0]. Note that the latter transformation is more radical reducing to zero all the membership values that are lower than the assumed threshold level. The higher the threshold level, the more significant the deformation of the original fuzzy set.

The second operation of interest in this method applies to the output fuzzy sets (**y**) and is introduced as a continuous mapping

$$\psi: [0,1] \rightarrow [0, 1]$$

such that

- ψ is increasing
- $\psi(1) = 1$
- $\psi(u) \geq u$

In comparison to the previous operation, the graph of ψ lies above the main diagonal of the unit square. Subsequently, the application of this mapping to any fuzzy set makes the resulting fuzzy set $\psi(\mathbf{y})$ less specific than its original counterpart. Two examples of the transforming functions belonging to this class are visualized in Fig. 3. These are defined explicitly as

$$\psi(u) = (1 - \beta)\, u + \beta \qquad (10)$$

as well as

$$\psi(u) = \begin{cases} \beta, \text{ if } u < \beta \\ u, \text{ if } u \geq \beta \end{cases} \qquad (11)$$

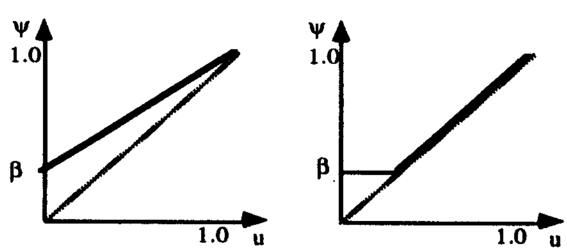

Fig.3. Two examples of specificity - decreasing transformations

Once the input output data have been transformed, the calculations of the fuzzy relation are carried out as described by (3). If, originally, (**x**(k), **y**(k)) are very difficult to interpolate and the theoretical solution could eventually lead to a fairly meaningless fuzzy relation, then the use of the modified data makes the entire procedure more feasible and realistic. This *specificity shift* paradigm that deals with the increased specificity of the input and decreased specificity of the output data is also intuitively plausible. By increasing and/or decreasing the specificity of the input and output data, respectively, we ease the complexity of the relational constraints to be interpolated. To look at this issue in more depth, let us consider two fuzzy relational equations

(two relational constraints)

$$x(1) \square R = y(1)$$
$$x(2) \square R = y(2)$$

assuming additionally that each of them is solvable individually. Obviously, this type of solvability does not imply that these two equations are solvable as a tandem. Observe, however, that if **x**(1) and **x**(2) are made disjoint then the ensuing system of equations is still solvable. Subsequently, to disjoint **x**(1) and **x**(2), one has to make them more specific - this is, in fact, the role of the first mapping.

Let us discuss a situation where $y_1 \neq y_2$ while at the same time x_1 and x_2 are very similar. No solution to the system of such equations exists yet the problem could be significantly reduced by making y_1 and y_2 very similar; such a similarity enhancement is accomplished through the decrease of their specificity.

4. Solving fuzzy relational equations via specificity shift of interpolation constraints

The algorithm combines the theory of the fuzzy relational equations with the heuristics of specificity shift applied to the input - output data (relational constraints). Let us briefly summarize the procedure:

1. select cutoff parameters of φ and ψ
2. transform data into a series of pairs $(\varphi(x(k)), \psi(y(k))$
3. compute the fuzzy relation with the use of (3)
4. verify the quality of the solution e.g., by calculating a sum of squared distances (MSE criterion) between **y**(k) and **x**(k) $\square$ R.

The entire process can be iterated with respect to the values of the cutoff parameters and these could be optimized so that they imply a minimal value of the MSE criterion (as outlined in the last phase of the above scheme).

Additionally, the obtained fuzzy relation can be viewed as a sound starting point for any finer optimization techniques, especially those relying on gradient - based mechanisms. Instead of being initialized from random fuzzy relations, one can start off the method from the fuzzy relation already computed in the first phase.

5. Concluding remarks

The specificity shift method can be classified as an approach situated in-between analytical and numerical methods of solving fuzzy relational equations. It relies on the original structure of the solution originating from the theory and simultaneously takes advantage of some optimization mechanisms available in the format of the parametric specificity shift affecting the relational constraints forming the fuzzy relational equations to be solved. In this sense the optimal threshold values of the transformation functions provide with a better insight into the character of the data to be handled especially when it comes to their overall consistency level. Especially, if the threshold values are high, these could serve as a sound indicator of a significant level of noise associated with the data; in the sequel one should not expect a superb quality of the fuzzy relation, no matter which optimization method will be pursued.

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