The Category of Fuzzy Rings

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Abstract: In this pater, the concepts of the homomorphisms of fuzzy rings and the category of fuzzy rings are introduced. Then the proporties of special subcategories of category of fuzzy rings are discussed.

Keywords: Fuzzy subring, Homomorphism of fuzzy rings, Category of fuzzy rings, Subcategory.

1 Homomorphisms of fuzzy rings

In this pater, R denotes a ring, L, L_1 and L_2 are all complete distributive lattice with 0 and 1. An L-fuzzy subset of R is a function $A: R \to L$. Let L[X] denote the set of whole L-fuzzy subsets of X.

Definition 1.1 The L-fuzzy subset A of ring R is called an L-fuzzy subring of R, if for any $x, y \in R$, the following requiements are met:

(1) $A(x-y) \geqslant A(x) \wedge A(y)$,

(2) $A(xy) \geqslant A(x) \wedge A(y)$.

If the condition (2) is replaced by $A(xy) \geqslant A(x) \vee A(y)$, then A is called an L-fuzzy ideal of R.

Let FL(R) denote the set of whole L-fuzzy subrings of R.

Definition 1.2 Let R_1 and R_2 be two rings, $A \in FL_1(R_1), B \in FL_2(R_2)$, an F-homomorphism of A into B is a mapping pair (f, τ) which satisfies;

(1) $f: R_1 \to R_2$ is a homomorphism of rings,

(2) $\tau: L_1 \to L_2$ holds arbitrary intersection and union properties, and $\tau(0) \in L_2$,

(3) $A \leqslant \tau^{-1} \circ B \circ f$, where τ^{-1} is defined by:

$$\tau^{-1}(a) = \bigvee\{\beta : \tau(\beta) \leqslant a, \beta \in L_1\} \quad \text{for any } a \in L_2.$$

Definition 1.3 Let R_1 and R_2 be two rings, $A \in FL_1(R_1)$, $B \in FL_2(R_2)$, an FQ-homomorphism of A into B is a mapping pair (f, τ) which satisfies:

(1) $f: R_1 \to R_2$ is a homomorphism of rings,

(2) $\tau: L_1 \to L_2$ holds arbitrary intersection and union properties, and $\tau(0) \in L_2$,

(3) $A = \tau^{-1} \circ B \circ f$, where τ^{-1} is defined by:

$$\tau^{-1}(a) = \bigvee\{\beta : \tau(\beta) \leqslant a, \beta \in L_1\} \quad \text{for any } a \in L_2.$$

Theorem 1.4 Let $(f, \tau) \in Hom(A,B)$, $(g, \psi) \in Hom(B,C)$. Define "$\circ$" by

$$(g, \psi) \circ (f, \tau) = (g \circ f, \psi \circ \tau)$$

then $(g \circ f, \psi \circ \tau) \in Hom(A,C)$ and "$\circ$" satisfies associative law where $A \in FL_1(R_1)$, $B \in FL_2(R_2)$, $C \in FL_3(R_3)$, R_1, R_2 and R_3 are rings.

Proof. It is quite evident that $g \circ f$ is a homomorphism of rings and $\phi \circ \tau$ holds arbitrary intersection and union properties. By Theorem 1.1 of [1], we have $(\psi \circ \tau)^{-1} = \tau^{-1} \circ \psi^{-1}$. Thus

$$A \leqslant \tau^{-1} \circ B \circ f \leqslant \tau^{-1} \circ \psi^{-1} \circ C \circ g \circ f = (\psi \circ \tau)^{-1} \circ C \circ (g \circ f),$$

That is, $(g, \psi) \circ (f, \tau) \in Hom(A,C)$.

Obviously, the associative law holds.

For FQ-homomorphism, we have the property which similar to Theorem 1.4.

Proposition 1.5 Let $B \in FL_2(R_2)$, $A \in L_1[R_1]$. If mapping pair (f, τ) satisfies condition in Definition 1.3, then τ^{-1} holds the arbitrary intersection and union properties and $A \in FL_1(R_1)$.

Proof. By the Proposition 1.1 in [1], we have that τ^{-1} holds the arbitrary intersection and union properties.

Morever, since $A = \tau^{-1} \circ B \circ f$ and $B \in FL_2(R_2)$, so

$$\begin{aligned} A(x_1 - x_2) &= (\tau^{-1} \circ B \circ f)(x_1 - x_2) \\ &= (\tau^{-1} \circ B)(f(x_1) - f(x_2)) \\ &\geqslant \tau^{-1}(B(f(x_1)) \wedge B(f(x_2))) \\ &= (\tau^{-1} \circ B \circ f)(x_1) \wedge (\tau^{-1} \circ B \circ f)(x_2) \end{aligned}$$

for any $x_1, x_2 \in R_1$.

Similarly, we have $A(x_1 x_2) \geqslant A(x_1) \wedge A(x_2)$, so, $A \in FL_1(R_1)$.

Proposition 1.6 Let $A\in FL_1(R_1)$, (f,τ) satisfies condition (1) and (2) in Definition 1.3, if $B\in L_2[R_2]$ is defined by

$$B(y) = \vee\{\tau(A(x)):x\in f^{-1}(y)\} \quad \text{for all } y\in R_2$$

then $B\in FL_2(R_2)$ and $\tau^{-1}\circ B\circ f \geqslant A$, we assume that the supremum of empty set is 0.

Proof. Let $y_1,y_2\in R_2$. If $B(y_1)=0$ or $B(y_2)=0$, we have $B(y_1-y_2)\geqslant B(y_1)\wedge B(y_2)$, otherwise $f^{-1}(y_1)\neq\phi$, and $f^{-1}(y_2)\neq\phi$, so $f^{-1}(y_1-y_2)\neq\phi$. Consequently,

$$\begin{aligned} B(y_1-y_2) &= \vee\{\tau(A(x)):x\in f^{-1}(y_1-y_2)\} \\ &\geqslant \vee\{\tau(A(x_1-x_2)):y_1=f(x_1),y_2=f(x_2)\} \\ &\geqslant \vee\{\tau(A(x_1))\wedge\tau(A(x_2)):y_1=f(x_1),y_2=f(x_2)\} \\ &= \vee\{\tau(A(x_1)):y_1=f(x_1)\}\wedge\vee\{\tau(A(x_2)):y_2=f(x_2)\} \\ &= B(y_1)\wedge B(y_2). \end{aligned}$$

Similarly, we have

$$B(y_1y_2) \geqslant B(y_1)\wedge B(y_2)$$

So, $B\in FL_2(R_2)$.

Besides, for any $x_1\in R_1$, we have

$$\tau^{-1}\circ B\circ f(x) = (\tau^{-1}\circ B)(f(x)) = \tau^{-1}(\vee\{\tau(A(x_1)):f(x)=f(x_1)\})$$
$$\geqslant\tau^{-1}(\tau(A(x_1))) = \vee\{\beta:\tau(\beta)\leqslant\tau(A(x_1)),\beta\in L_1\}\geqslant A(x).$$

Therefore $\tau^{-1}\circ B\circ f\geqslant A$.

2. Category of fuzzy rings and its subcategories

Definition 2.1 Category of fuzzy rings FR is defined by:

(1) Its objects are $\text{obj}FR=\{A\in FL(R): L$ is an arbitrary lattice, R is an arbitrary ring$\}$,

(2) For any A, $B\in \text{obj}FR$, the arrows from A into B is $\text{Hom}(A,B)=\{(f,\tau):(f,\tau)$ satisfies conditions (1)-(3) in Definition 1.2$\}$

(3) Let $(f,\tau)\in\text{Hom}(A,B)$, $(g,\psi)\in\text{Hom}(B,C)$, the product $(g,\psi)\circ(f,\tau)$ is defined by Theorem 1.4.

Evidently, the Definition 2.1 is reasonable.

Here our definition of category is in keeping with [4], that is, we give up condition: if $A\neq A'$ or $B\neq B'$, then $\text{Hom}(A,B)\cap\text{Hom}(A',B')=\phi$.

Definition 2.2 The categories FRL, $F[R']$, $FRL[\tau]$, FR_0, FRQ, FRQL, $FQ[R']$, $FRQL[\tau]$, FRQ_0 are defined by

(1) ObjFRL=$\{A\in FL(R)$: L is an arbitrary lattice, R is a fixed ring$\}$ the arrows are all F-homomorphisms,

(2) Obj$F[R']=\{A\in FL(R')$: L is an arbitrary lattice, R' is a fixed ring$\}$ the arrows are all F-homomorphisms,

(3) ObjFRL$[\tau]$=ObjFRL, the arrows are all F-homomorphisms (f,τ), where τ is a fixed mapping,

(4) $FR_0 = FR_{\{1,0\}}$,

(5) ObjFRQ=$\{A\in FL(R)$: R is an arbitrary ring, L is an arbitrary lattice$\}$, the arrows are all F-homomorphisms,

(6) ObjFRQL = ObjFRL, the arrows are all FQ-homomorphisms,

(7) ObjFRQ$[R']$ = Obj$F[R']$, the arrows are all FQ-homomorphisms,

(8) ObjFRQL$[\tau]$ = ObjFRL$[\tau]$, the arrows are all FQ-homomorphisms (f,τ), where τ is a fixed mapping,

(9) $FRQ_0 = FRQ_{\{1,0\}}$.

The following Proposition 2.3 is the consequence of Definition 2.2 .

Proposition 2.3 (1) FRL and $FR[R']$ are full subcategory of FR,

(2) FRQL and $FQ[R']$ are full subcategory of FRQ.

Let A, B $\in$ ObjFRQ and

$Hom_Q(A,B)=\{(f,\tau);(f,\tau)$ satisfies conditions (1)-(3) in Definition 1.3$\}$

Then we have the following Theorem 2.4.

Theorem 2.4 Let $L=\{0,1\}$, $A\in FL(R_1)$, $B\in FL(R_2)$, then the arrows set of FRQL = FRQ_0 satisfies

(1) If $A\neq R_1$, then $Hom_Q(A,B)=\phi$ or $Hom_Q(A,B)=\{(f,\tau)$: f is a homomorphism of modules, $\tau=i_L$, $A=B\circ f\}$, where i_L is an unit mapping on L.

(2) If there is $(f,0)\in Hom(A,B)$, then $A=R_1$.

Proof. It is similar to Theorem 2.1 of [1], and hence ommitted.

Theorem 2.5 Let $L=\{0,1\}$, then ordinary ring category and category FRQ_0 are isomorphic.

Proof. It is easy, and hence ommitted.

Theorem 2.6 $FRL[\tau]$ is a subcategory of FRL, but is not a full subcategory of FRL.

Proof. It is similar to the proof of Theorem 2.3 in [1] and hence ommitted.

Theorem 2.7 Let L_1 and L_2 be isomorphic lattices, R_1 and R_2 be two isomorphic rings, then

(1) FRL_1 and FRL_2 are isomorphic categories.

(2) $FRQL_1$ and $FRQL_2$ are isomorphic categories.

(3) $F[R_1]$ and $F[R_2]$ are isomorphic categories.

(4) $FQ[R_1]$ and $FQ[R_2]$ are isomorphic categories.

Proof. (1) Let $\alpha: L_1 \to L_2$ is an isomorphism of lattices, then $\alpha \circ \alpha^{-1} = i_L$ and $\alpha^{-1} \circ \alpha = i_L$. Dfine functor $\theta: FRL_1 \to FRL_2$, such that $\theta(A) = \theta \circ A$, $\theta(f, \tau) = (f, \theta \circ \tau \circ \theta^{-1})$, for all $A, B \in Obj(FRL_1)$, $(f, \tau) \in Hom(A, B)$. Similar to Theorem 2.4 of [1], we can prove θ to be a isomorphic map from FRL_1 to FRL_2.

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