LATTICE OF FUZZY CONGRUENCES IN INVERSE SEMIGROUPS

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1. INTRODUCTION:

Fuzzy relations have been investigated by several authors. In particular, Sanchez [8], Murali [4], Nemitz [6] etc. have made significant contribution in this field. The fuzzy analogues of congruence relations which play an important role in getting quotient structures have been studied by Murali [5] in the case of universal algebra, Kuroki [3] in the case of groups and more recently by Al Thukair [1] in the case of inverse semigroups.

In continuation of the study undertaken by Al Thukair [1], the fuzzy analogue of the lattice of congruences in an inverse semigroup is developed in the present paper. In Section 3, the lattice structure of fuzzy congruences is introduced in an arbitrary semigroup. For an arbitrary congruence relation ρ, the fuzzy congruence relations ρ_{min} and ρ_{max} are defined and some interesting results concerning them are obtained. An important result (Theorem 3.11) concerning the lattice of fuzzy congruences in relation to their fuzzy traces is proved. Theorem 3.15 provides a clear picture of the close relation between a fuzzy congruence and its fuzzy trace and kernel.

2. PRELIMINARIES:

Let I denote the closed unit interval [0,1] . Let X be a non-empty set . M(x,y) will denote the minimum of x and y .

A fuzzy relation ρ on X is a mapping $\rho : X\times X \longrightarrow I$. If ρ and λ be two fuzzy relations on X , $\rho \le \lambda$ means that $\rho(x,y) \le \lambda(x,y)$ for all $x,y \in X$. The composition of two fuzzy relations ρ and λ on X is defined by

$(\rho\ \lambda)(x,y) = \sup \{ M(\rho(x,z) ,\lambda(z,y)) ; z \in X \}$ for all $x,y \in X$.

$\rho\cap\lambda$ and $\rho\cup\lambda$ will denote the fuzzy intersection and union of ρ and λ .

Let S be a semigroup .

DEFINITION 2.1 [4]: A fuzzy relation ρ on S is called a fuzzy equivalence relation if

(i) $\rho(x,x) = 1$ for all $x \in S$.

(ii) $\rho(x,y) = \rho(y,x)$ for all $x,y \in S$.

(iii) $\rho \ge \rho_0\rho$.

DEFINITION 2.2 [1]: A fuzzy relation ρ on S is called fuzzy left (resp. right) compatible , if $\rho(sx,sy)$ (resp. $\rho(xs,ys)$) $\ge \rho(x,y)$ for all $x,y,s \in S$.

DEFINITION 2.3 [1]: A fuzzy equivalence relation ρ on S is called a fuzzy congruence relation if

$$M(\rho(x,y) ,\rho(z,w)) \le \rho(xz.yw) \text{ for all } x,y,z,w \in S .$$

LEMMA 2.4 [1]: A fuzzy equivalence relation ρ is a fuzzy congruence relation iff it is both fuzzy left and right compatible .

Let S be an inverse semigroup . Let E(S) denote the set of all idempotents of S .

DEFINITION 2.5 [1]: A fuzzy congruence relation ρ on

E(S) is called a fuzzy normal congruence relation if $\rho(s^{-1}es, s^{-1}fs) \geq \rho(e,f)$ for all $e,f \in E(S)$, for all $s \in S$.

DEFINITION 2.6 [1]: Let ρ be a fuzzy congruence relation on S . The fuzzy kernel of ρ , denoted by $K(\rho)$, is defined by $K(\rho)(x) = \sup \{ \rho(x.e) ; e \in E(S) \}$ for all $x \in S$. The fuzzy trace of ρ , denoted by $T(\rho)$, is defined by $T(\rho)(e,f) = \rho(e,f)$ for all $e,f \in E(S)$.

It can be easily verified that $T(\rho)$ is a fuzzy normal congruence relation on E(S) .

LEMMA 2.7 : If ρ is a fuzzy congruence relation on S , then $\rho(x,y) = \rho(x^{-1},y^{-1})$ for all $x,y \in S$.

Henceforth unless otherwise stated , we follow Howie [2] and Petrich [7] for the notions and notations of semigroup .

3. LATTICE OF FUZZY CONGRUENCES IN INVERSE SEMIGROUP

We first develop the theory of the lattice of fuzzy congruences in an arbitrary semigroup which will be needed latter .

Let S be a semigroup . Let C(S) denote the set of all fuzzy congruence relations on S . It is partially ordered by the inclusion relation $\leq$. Now for any two fuzzy congruence relations ρ and σ on S , $\rho\cap\sigma$ is the greatest lower bound of ρ and σ . Since $\rho\cup\sigma$ is not necessarily fuzzy transitive , we take the fuzzy transitive closure of $\rho\cup\sigma$ and denote it by $\rho\vee\sigma$. Thus $\rho\vee\sigma = \sup \{ (\rho\cup\sigma)^n ; n \in N \}$. Clearly $\rho\vee\sigma$ is a fuzzy congruence relation .

THEOREM 3.1 : (C(S) ,$\leq$,$\cap$,$\vee$) is a complete lattice .

THEOREM 3.2 : For any two fuzzy congruence relations ρ, σ on S , $\rho\vee\sigma = \sup \{ (\rho\circ\sigma)^n ; n \in N \}$.

COROLLARY 3.3 : Let ρ and σ be two fuzzy congruence relations on S , such that $\rho\circ\sigma = \sigma\circ\rho$. Then $\rho\vee\sigma = \rho\circ\sigma$.

THEOREM 3.4 : If H be a sublattice of the lattice (C(S) ,≤ ,∩ ,∨) of fuzzy congruences on S , such that $\rho\circ\sigma = \sigma\circ\rho$ for all $\rho,\sigma \in H$, then H is a modular lattice .

Let S be an inverse semigroup and let E(S) denote the set of all idempotents of S .

THEOREM 3.5 : If ρ ,σ be two fuzzy normal congruence relations on E(S) , then $\rho\vee\sigma$ is a fuzzy normal congruence relation .

Let H(E(S)) denote the set of all fuzzy normal congruence relations on E(S) .

THEOREM 3.6 : (H(E(S)) ,≤,∩ ,∨) is a complete lattice.

THEOREM 3.7 : Let ρ be a fuzzy congruence relation on E(S) . Then the fuzzy relation τ on E(S) defined by

$$\tau(e,f) = \inf \{ \rho(a^{-1}ea,a^{-1}fa) ; a \in S \}$$

is the greatest fuzzy normal congruence relation on E(S) contained in ρ .

THEOREM 3.8 : Let ρ be a fuzzy congruence relation on S . Then the fuzzy relation ρ_{max} on S defined by

$\rho_{max}(a,b) = \inf \{ \rho(a^{-1}ea,b^{-1}eb) ; e \in E(S) \}$ for all $a,b \in S$ is a fuzzy congruence relation on S .

We use the following lemma to prove the next theorem .

LEMMA 3.9 : A fuzzy relation ρ on S is fuzzy transitive iff $\rho_{st} = \{ (x,y) \in SxS ; \rho(x,y) > t \}$ is transitive relation for all $t \in [0,1)$.

THEOREM 3.10 : For any fuzzy congruence relation ρ on S , the fuzzy relation ρ_{min} on S defined by

$\rho_{min}(a,b)$ = sup { $M(\rho(e,a^{-1}a), \rho(a^{-1}a,b^{-1}b))$; e $\in$ E(S) such that ae = be } , if there exists e $\in$ E(S) such that ae = be , and $\rho_{min}(a,b) = \emptyset$, otherwise

is a fuzzy congruence relation on S .

THEOREM 3.11 : The mapping θ : C(S) ——> H(E(S)) defined by $\theta(\rho) = T(\rho)$, $\rho \in C(S)$ is a complete homomorphism of C(S) onto H(E(S)) . If R is the congruence on C(S) induced by θ , then $\rho R = [\rho_{min} ; \rho_{max}]$ for $\rho \in C(S)$ where ρR is the congruence class of R containing ρ , and ρR is a complete modular sublattice of (C(S) ,$\leq$,$\cap$,$\vee$) .

THEOREM 3.12 : Let ρ ,$\sigma \in C(S)$ be such that $\rho \leq \sigma$. Then $\rho_{min} \leq \sigma_{min}$ and $\rho_{max} \leq \sigma_{max}$.

THEOREM 3.13 : Let G be a nonempty family of fuzzy congruence relations on S . Then

(i) $\vee(\rho_{min} ; \rho \in G) = (\vee(\rho ; \rho \in G))_{min}$

(ii) $\cap(\rho_{max} ; \rho \in G) = (\cap(\rho ; \rho \in G))_{max}$.

The next theorem gives a complete characterisation of the fuzzy kernel of the relations ρ_{min} and ρ_{max} .

THEOREM 3.14 : For any $\rho \in C(S)$,

(i) $K(\rho_{max})(a)$ = inf { $\rho(ae,ea)$; e $\in$ E(S) } for all a $\in$ S

(ii) $K(\rho_{min}) = \lambda$, where λ: S ——>I is defined by

$\lambda(a)$ = sup { $\rho(e,a^{-1}a)$; e $\in$ E(S) such that ae = e } , if there exists e $\in$ E(S) such that ae = e .

$\lambda(a) = \emptyset$, otherwise .

THEOREM 3.15 : For any $\rho \in C(S)$,

$\rho_e = K(\rho) \cap (\rho_{max})_e = K(\rho) \cap ((T(\rho))_{max})_e$ for all $e \in E(S)$.

The theorem 3.15 shows that every fuzzy class of S determined by ρ and e ($e \in E(S)$) is completely determined by its fuzzy trace and kernel and this explicitly demonstrates the close connection between the fuzzy trace and kernel with the corresponding fuzzy congruence relation .

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