On Fuzzy Separation Axioms in Intuitionistic Fuzzy Topological Spaces

Sadık Bayhan[(1)]-Doğan Çoker[(2)]

[(1)] *Department of Mathematics, Hacettepe University, Beytepe, 06532-Ankara/TURKEY*

[(2)] *Department of Mathematics Education, Hacettepe University, Beytepe, 06532-Ankara/TURKEY*

Abstract: The purpose of this paper is to investigate several types of separation axioms in intuitionistic fuzzy topological spaces, developed by Çoker and et al. [5,6]. After giving some characterizations of separation axioms T_1 and T_2 in intuitionistic fuzzy topological spaces, interrelations between several types of separation axioms and some necessary counterexamples will be given.

Keywords: Intuitionistic fuzzy set; intuitionistic fuzzy topology; intuitionistic fuzzy topological space; intuitionistic fuzzy pair; fuzzy separation.

1. Introduction

After the introduction of the concept of a fuzzy set by Zadeh [17], Atanassov [2,3] has introduced the concept of intuitionistic fuzzy set (IFS for short). Çoker [5,6] has defined intuitionistic fuzzy topological spaces (IFTS's for short).

2. Preliminaries

For the purpose of completeness, we shall give some introductory definitions first:

Definition 2.1. [2,3] Let X be a nonempty fixed set. An intuitionistic fuzzy set (IFS for short) A is an object having the form

$$A=\{ \langle x,\mu_A(x),\gamma_A(x)\rangle : x\in X \}$$

where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x\in X$ to the set A, respectively, and $0\leq \mu_A(x)+\gamma_A(x)\leq 1$ for each $x\in X$. For the sake of simplicity, we shall use the symbol $A=\langle x,\mu_A,\gamma_A\rangle$ for the IFS $A=\{\langle x,\mu_A(x),\gamma_A(x)\rangle: x\in X\}$.

Correspondence to : Sadık Bayhan, Matematik Bölümü, Fen Fakültesi, Hacettepe Üniversitesi, Beytepe, 06532-Ankara / TURKEY

[e-Mail : bayhan@eti.cc.hun.edu.tr]

Every fuzzy set A on a nonempty set X is obviously an IFS having the form $A=\{ \langle x,\mu_A(x),1-\mu_A(x)\rangle : x\in X \}$ [3].

Definition 2.2. [3,5,6] Let X be a nonempty set, and the IFS's A and B be in the form $A=\{\langle x,\mu_A(x),\gamma_A(x)\rangle:x\in X\}$, $B=\{\langle x,\mu_B(x),\gamma_B(x)\rangle:x\in X\}$, and let $\{A_i:i\in J\}$ be an arbitrary family of IFS's in X. Then

(a) $A\subseteq B$ iff $\mu_A(x)\le\mu_B(x)$ and $\gamma_A(x)\ge\gamma_B(x)$ for all $x\in X$;
(b) $A=B$ iff $A\subseteq B$ and $B\subseteq A$;
(c) $\bar{A}=\{ \langle x,\gamma_A(x),\mu_A(x)\rangle : x\in X \}$;
(d) $\cap A_i=\{ \langle x,\wedge\mu_{A_i}(x),\vee\gamma_{A_i}(x)\rangle : x\in X \}$;
(e) $\cup A_i=\{ \langle x,\vee\mu_{A_i}(x),\wedge\gamma_{A_i}(x)\rangle : x\in X \}$;
(f) $0_\sim=\{ \langle x,0,1\rangle : x\in X \}$ and $1_\sim=\{ \langle x,1,0\rangle : x\in X \}$.

Using the definition of fuzzy topological spaces given by Chang [4], intuitionistic fuzzy topological spaces can be defined as follows:

Definition 2.3. [5,6] An intuitionistic fuzzy topology (IFT for short) on a nonempty set X is a family τ of IFS's in X containing $0_\sim$, $1_\sim\in\tau$, and closed under finite infima and arbitrary suprema. In this case the pair (X,τ) is called an intuitionistic fuzzy topological space (IFTS for short) and any IFS in τ is known as an intuitionistic fuzzy open set (IFOS for short) in X. The complement $\bar{A}$ of an IFOS A in an IFTS (X,τ) is called an intuitionistic fuzzy closed set (IFCS for short) in X.

Example 2.4. [5,6] Any fuzzy topological space (X,τ_o) in the sense of Chang is obviously an IFTS in the form $\tau=\{A:\mu_A\in\tau_o\}$ whenever we identify a fuzzy set in X whose membership function is μ_A with its counterpart $A=\{\langle x,\mu_A(x),1-\mu_A(x)\rangle:x\in X\}$.

Definition 2.5. [5,6] Let A be an IFS in (X,τ). Then

$$cl(A)=\cap\{K : K \text{ is an IFCS in } X \text{ and } A\subseteq K\},$$
$$int(A)=\cup\{G : G \text{ is an IFOS in } X \text{ and } G\subseteq A\}.$$

Definition 2.6. [7] Let X be a nonemptyset and $c\in X$ a fixed element in X. If $\alpha\in(0,1]$ and $\beta\in[0,1)$ are two real numbers such that $\alpha+\beta\leq 1$, then

(a) $c(\alpha,\beta)=<x,c_\alpha,1-c_{1-\beta}>$ is called an intuitionistic fuzzy point (IFP for short) in X, where α denotes the degree of membership of $c(\alpha,\beta)$, β the degree of nonmembership of $c(\alpha,\beta)$.

(b) $c(\beta)=<x,0,1-c_{1-\beta}>$ is called a vanishing intuitionistic fuzzy point (VIFP for short) in X, where β denotes the degree of nonmembership of $c(\beta)$.

Definition 2.7. [7] (a) Let $c(\alpha,\beta)$ be an IFP in X. and $A=<x,\mu_A,\gamma_A>$ an IFS in X. $c(\alpha,\beta)\subseteq A$ is said to be contained in A, ($c(\alpha,\beta)\leq A$ for short) iff $c(\alpha,\beta)\subseteq A$. [In other words, $c(\alpha,\beta)\leq A$ iff $c_\alpha\leq\mu_A$ and $1-c_{1-\beta}\geq\gamma_A$, or equivalently, $\alpha\leq\mu_A(c)$ and $\beta\geq\gamma_A(c)$.]

(b) Let $c(\beta)$ be a VIFP in X and $A=<x,\mu_A,\gamma_A>$ an IFS in X. $c(\beta)$ is said to be contained in A ($c(\beta)\leq A$ for short) iff $\mu_A(c)=0$ and $1-c_{1-\beta}\geq\gamma_A$, or equivalently, $\mu_A(c)=0$ and $\beta\geq\gamma_A(c)$.)

Products of intuitionistic fuzzy topological spaces

Let (X,τ) and (Y,Φ) be two IFTS's, and $A\in I^X$, $B\in I^Y$. Then the product of A and B is defined b as in [3] by

$$AxB = \{ <(x,y),\mu_A(x)\wedge\mu_B(y),\gamma_A(x)\vee\gamma_B(y)> : (x,y)\in XxY \}$$

Now we can construct the product topology on XxY as the initial IFT on XxY with respect to the projections

$$\pi_1 : XxY \to X,\ \pi_1(x,y)=x \quad\text{and}\quad \pi_2 : XxY \to Y,\ \pi_2(x,y)=y.$$

In this case, the subbase of the product IFT is given by

$$\mathcal{S} = \{ \pi_1^{-1}(T_1),\ \pi_2^{-1}(T_2) : T_1\in\tau,\ T_2\in\Phi \}.$$

Hence the base generated by $\mathcal{S}$ can be written as

$$\mathcal{B} = \{ \pi_1^{-1}(T_1)\cap\pi_2^{-1}(T_2) : T_1\in\tau,\ T_2\in\Phi \}.$$

Since

$$\pi_1^{-1}(T_1)\cap\pi_2^{-1}(T_2)=<(x,y),\mu_{T_1}(x)\wedge\mu_{T_2}(y),\gamma_{T_1}(x)\vee\gamma_{T_2}(y)>=T_1xT_2,$$

we easily obtain $\mathcal{B} = \{ T_1xT_2 : T_1\in\tau,\ T_2\in\Phi \}$.

Definition 2.8. Given the nonempty set X, we define the diagonal Δ as the following IFS in XxX:

$$\Delta = \langle (x_1,x_2),\ \mu_\Delta(x_1,x_2),\ \gamma_\Delta(x_1,x_2) \rangle,$$

where

$$\mu_\Delta(x_1,x_2)=\begin{cases} 1, & \text{if } x_1=x_2 \\ 0, & \text{if } x_1\neq x_2 \end{cases} \quad \text{and} \quad \gamma_\Delta(x_1,x_2)=\begin{cases} 0, & \text{if } x_1=x_2 \\ 1, & \text{if } x_1\neq x_2 \end{cases}.$$

3. T_1 and T_2 properties in IFTS's

First, we introduce the concept of intuitionistic fuzzy pair:

Definition 3.1. [8] Let a and b be two real numbers in [0,1] satisfying the inequality $a+b\leq 1$. Then the pair $\langle a,b\rangle$ is called an intuitionistic fuzzy pair.

Definition 3.2. [8] Let $\langle a,b\rangle$, $\langle a_1,b_1\rangle$, $\langle a_2,b_2\rangle$, $\langle a_i,b_i\rangle$ $(i\in J)$ be intuitionistic fuzzy pairs. Then we define

(a) $\langle a_1,b_1\rangle\leq\langle a_2,b_2\rangle \iff a_1\leq a_2$ and $b_1\geq b_2$,

(b) $\langle a_1,b_1\rangle=\langle a_2,b_2\rangle \iff a_1=a_2$ and $b_1=b_2$,

(c) $\vee\langle a_i,b_i\rangle=\langle\vee a_i,\wedge b_i\rangle$ and $\wedge\langle a_i,b_i\rangle=\langle\wedge a_i,\vee b_i\rangle$;

(d) $\overline{\langle a,b\rangle}=\langle b,a\rangle$ (the complement of $\langle a,b\rangle$) ;

(e) $\tilde{1} = \langle 1,0\rangle$ and $\tilde{0} = \langle 0,1\rangle$.

Now we shall first define several fuzzy T_1-separation axioms:

Definition 3.3. Let (X,τ) be an IFTS.

(1): (X,τ) is called $FT_1(i)$ iff (a) for each pair of distinct IFP's $x(\alpha,\beta)$ and $y(\xi,\eta)$ in X, there exist $U,V\in\tau$ such that $x(\alpha,\beta)\subseteq U$, $y(\xi,\eta)\not\subseteq U$ and $y(\xi,\eta)\subseteq V$, $x(\alpha,\beta)\not\subseteq V$.
[cf. Srivastava-Lal-Srivastava [15]]
(b) for each pair of distinct VIFP's $x(\beta)$ and $y(\eta)$ in X, there exist $U,V\in\tau$ such that $x(\beta)\subseteq U$, $y(\eta)\not\subseteq U$ and $y(\eta)\subseteq V$, $x(\beta)\not\subseteq V$.

(2): (X,τ) is called $FT_1(ii)$ iff for all $x,y\in X$, $x\neq y$, there exist $U,V\in\tau$ such that $U(x)=\tilde{1}$, $U(y)=\tilde{0}$ and $V(y)=\tilde{1}$, $V(x)=\tilde{0}$.
[cf. Srivastava-Lal-Srivastava [16]]

(3): (X,τ) is called $FT_1(iii)$ iff (a) for each pair of distinct IFP's $x(\alpha,\beta)$ and $y(\xi,\eta)$ in X, there exist $U,V\in\tau$ such that
$x(\alpha,\beta)\subseteq U\subseteq\overline{y(\xi,\eta)}$ and $y(\xi,\eta)\subseteq V\subseteq\overline{x(\alpha,\beta)}$

[cf. Ghanim-Kerre-Mashhour [11]]

(b) for each pair of distinct VIFP's $x(\beta)$ and $y(\eta)$ in X, there exist $U,V\in\tau$ such that $x(\beta)\subseteq U\subseteq\overline{y(\eta)}$ and $y(\eta)\subseteq V\subseteq\overline{x(\beta)}$.

(4): (X,τ) is called $FT_1(iv)$ iff (a) for each pair of distinct IFP's $x(\alpha,\beta)$ and $y(\xi,\eta)$ in X, there exist $U,V\in\tau$ such that $x(\alpha,\beta)\subseteq U$, $U\cap y(\xi,\eta)=\underset{\sim}{0}$ (i.e. $U(y)=\tilde{0}$) and $y(\xi,\eta)\subseteq V$, $V\cap x(\alpha,\beta)=\underset{\sim}{0}$ (i.e. $V(x)=\tilde{0}$) .

[cf. Fora [9]]

(b) for each pair of distinct VIFP's $x(\beta)$ and $y(\eta)$ in X, there exist $U,V\in\tau$ such that $x(\beta)\subseteq U$, $U(y)=\tilde{0}$ and $y(\eta)\subseteq V$, $V(x)=\tilde{0}$.

(5): (X,τ) is called $FT_1(v)$ iff (a) for each pair of distinct IFP's $x(\alpha,\beta)$ and $y(\xi,\eta)$ in X, there exist $U,V\in\tau$ such that $U(y)=\tilde{0}$ and $V(x)=\tilde{0}$.

[cf. Katsaras [12]]

(b) for each pair of distinct VIFP's $x(\beta)$ and $y(\eta)$ in X, there exist $U,V\in\tau$ such that $U(y)=\tilde{0}$ and $V(x)=\tilde{0}$.

(6): (X,τ) is called $FT_1(vi)$ $\Longleftrightarrow$ for all $x,y\in X$, $x\neq y$, there exist $U,V\in\tau$ such that $U(x)>U(y)$ and $V(y)>V(x)$.

[cf. Ali [1]]

Theorem 3.4. Let (X,τ) be an IFTS. Then the following implications are valid:

$$FT_1(ii) \Longleftrightarrow FT_1(iii) ;$$
$$FT_1(ii) \Longrightarrow FT_1(i) ;$$
$$FT_1(iv) \Longleftrightarrow FT_1(ii) ;$$
$$FT_1(iv) \Longleftrightarrow FT_1(v) ;$$
$$FT_1(ii) \Longrightarrow FT_1(vi) .$$

Proof. ($FT_1(ii) \Rightarrow FT_1(iii)$): (a) Let $x(\alpha,\beta)$ and $y(\xi,\eta)$ be two distinct IFP's in X. Then there exist $U,V\in\tau$ such that $U(x)=\tilde{1}$, $U(y)=\tilde{0}$ and $V(y)=\tilde{1}$, $V(x)=\tilde{0}$. Then we have $\alpha\leq 1=\mu_U(x)$, $\beta\geq 0=\gamma_U(x)$ and $\mu_U(y)=0\leq\eta$, $\gamma_U(y)=1\geq\xi$. It is clear that $x(\alpha,\beta)\subseteq U\subseteq\overline{y(\xi,\eta)}$. The other part can be shown similarly.

(b) Let $x(\beta)$, $y(\eta)$ be two distinct VIFP's in X. Then there exist $U,V\in\tau$ such that $U(x)=\tilde{1}$, $U(y)=\tilde{0}$ and $V(y)=\tilde{1}$, $V(x)=\tilde{0}$. For $z\neq x$,

$0\le\mu_U(z)$, $1\ge\gamma_U(z)$ and for $z=x$, $0\le\mu_U(x)$, $\beta\ge\gamma_U(x)=0$. Then we have $x(\beta)=\langle z,0,1-x_{1-\beta}\rangle\subseteq U$. For $z\neq y$, $\mu_U(z)\le 1$, $\gamma_U(z)\ge 0$ and for $z=y$, $0=\mu_U(y)\le\eta$, $\gamma_U(y)\ge 0$. Then we have $U\subseteq\langle z,1-y_{1-\eta},0\rangle=\overline{y(\eta)}$. Thus we conclude that $x(\beta)\subseteq U\subseteq\overline{y(\eta)}$, and similarly, $y(\eta)\subseteq V\subseteq\overline{x(\beta)}$.

(FT_1(iii) $\Rightarrow$ FT_1(ii)): (a) Let $x\neq y$ and consider the IFP's $x(1,0)$ and $y(1,0)$ in X. Then there exist $U,V\in\tau$ such that $x(1,0)\subseteq U\subseteq\overline{y(1,0)}$ and $y(1,0)\subseteq V\subseteq\overline{x(1,0)}$. The inclusions $x(1,0)\subseteq U$ and $y(1,0)\subseteq V$ imply $U(x)=\tilde{1}$ and $V(y)=\tilde{1}$. On the other hand, $U(x)\subseteq\overline{y(1,0)}=\langle z,1-y_1,y_1\rangle$ implies that, for $z\neq y$, $\mu_U(z)\le 1$, $\gamma_U(z)\ge 0$ and for $z=y$, $\mu_U(y)=0\le\eta$, $\gamma_U(y)\ge 0$. Then we have $U(y)=\tilde{0}$. It can be shown similarly that $V(x)=\tilde{0}$.

(b) Let $x\neq y$, and consider the VIFP's $x(0)$ and $y(0)$ in X. Then there exist $U,V\in\tau$ such that $x(0)\subseteq U\subseteq\overline{y(0)}$ and $y(0)\subseteq V\subseteq\overline{x(0)}$, from which $U(x)=\tilde{1}$, $U(y)=\tilde{0}$ and $V(y)=\tilde{1}$, $V(x)=\tilde{0}$ follow.

(FT_1(ii) $\Rightarrow$ FT_1(i)): (a) Let $x\neq y$, and $x(\alpha,\beta)$, $y(\xi,\eta)$ be two distinct IFP's in X. Then there exist $U,V\in\tau$ such that $U(x)=\tilde{1}$, $U(y)=\tilde{0}$ and $V(y)=\tilde{1}$, $V(x)=\tilde{0}$. Then we have $\langle\alpha,\beta\rangle\le\tilde{1}=U(x)$, $x(\alpha,\beta)\subseteq U$ and $\langle\xi,\eta\rangle\le\tilde{1}=V(y)$, $y(\xi,\eta)\subseteq V$. $U(y)=\tilde{0}\Longrightarrow\mu_U(y)=0$, $\gamma_U(y)=1$ $\Longrightarrow$ $y(\xi,\eta)\not\subseteq U$. $V(x)=\tilde{0}$ implies that $\mu_V(x)=0$, $\gamma_V(x)=1$, and then $x(\alpha,\beta)\not\subseteq V$ follows. Hence FT_1(i) is true.

(b) Let $x(\beta)$, $y(\eta)$ be two distinct VIFP's in X. Then there exist $U,V\in\tau$ such that $U(x)=\tilde{1}$, $U(y)=\tilde{0}$ and $V(y)=\tilde{1}$, $V(x)=\tilde{0}$. For $z\neq x$, $0\le\mu_U(z)$, $1\ge\gamma_U(z)$ and for $z=x$, $\mu_U(x)=1$, $\beta\ge\gamma_U(x)=0$. Then we have $\langle z,0,1-x_{1-\beta}\rangle\subseteq U$. On the other hand, for $z\neq y$, $0\le\mu_U(z)$, $1\ge\gamma_U(z)$ and for $z=y$, $0=\mu_U(y)$, $\gamma_U(y)=1$. But $\eta\ge\gamma_U(y)=1$ does not hold, since $\eta<1$. Hence $\langle z,0,1-y_{1-\eta}\rangle=y(\eta)\not\subseteq U$. $y(\eta)\subseteq V$ and $x(\beta)\not\subseteq V$ can be proved similarly. ■

(FT_1(iv) $\Longleftrightarrow$ FT_1(ii)),(FT_1(iv) $\Longleftrightarrow$ FT_1(v)) and (FT_1(ii) $\Rightarrow$ FT_1(vi)): They are obvious. ■

Counterexample 3.5. Let X={a,b} and define the IFS's U,V as follows:

$U = \langle x, (0.5,0.4), (0.3,0.4) \rangle,$

$V = \langle x, (0.2,0.3), (0.7,0.6) \rangle.$

The family $\tau=\{\underset{\sim}{0},\underset{\sim}{1},U,V,U\cap V,U\cup V\}$ is an IFT on X. It is clear that (X,τ) is $FT_1(vi)$, but it is not $FT_1(ii)$. Because there are not IFOS's in (X,τ) which has the properties $U(x)=\tilde{1}$, $U(y)=\tilde{0}$ and $V(y)=\tilde{1}$, $V(x)=\tilde{0}$.

Definition 3.6. Let (X,τ) be an IFTS.

(1): (X,τ) is called $FT_2(i)$ iff for all $x,y\in X$, $x\neq y$, there exist $U,V\in\tau$ such that $U(x)=\tilde{1}$, $V(y)=\tilde{1}$ and $U\cap V=\underset{\sim}{0}$.

[cf. Gantner-Steinlage-Warren [10]]

(2): (X,τ) is called $FT_2(ii)$ iff (a) for each pair of distinct IFP's $x(\alpha,\beta)$ and $y(\xi,\eta)$ in X, $\exists$ $U,V\in\tau$: $x(\alpha,\beta)\subseteq U$, $y(\xi,\eta)\subseteq V$ and $U\cap V=\underset{\sim}{0}$.

[cf. Srivastava-Lal-Srivastava [14]]

(b) for each pair of distinct VIFP's $x(\beta)$ and $y(\eta)$ in X, there exist $U,V\in\tau$ such that $x(\beta)\subseteq U$, $y(\eta)\subseteq V$ and $U\cap V=\underset{\sim}{0}$.

(3): (X,τ) is called $FT_2(iii)$ iff for all $x,y\in X$, $x\neq y$, there exist $U,V\in\tau$ such that $U(x)\neq\tilde{0}$, $V(y)\neq\tilde{0}$ and $U\cap V=\underset{\sim}{0}$.

[cf. Katsaras [12]]

(4): (X,τ) is called $FT_2(iv)$ iff (a) for each pair of distinct IFP's $x(\alpha,\beta)$ and $y(\xi,\eta)$ in X, there exist $U,V\in\tau$ such that $x(\alpha,\beta)\subseteq U\subseteq\overline{y(\xi,\eta)}$, $y(\xi,\eta)\subseteq V\subseteq\overline{x(\alpha,\beta)}$ and $U\subseteq\overline{V}$.

[cf. Ghanim-Kerre-Mashhour [11]]

(b) for each pair of distinct VIFP's $x(\beta)$ and $y(\eta)$ in X, there exist $U,V\in\tau$ such that $x(\beta)\subseteq U\subseteq\overline{y(\eta)}$, $y(\eta)\subseteq V\subseteq\overline{x(\beta)}$ and $U\subseteq\overline{V}$.

(5): (X,τ) is called $FT_2(v)$ iff for all $x,y\in X$, $x\neq y$, there exist $U,V\in\tau$ such that $U(x)=\tilde{1}=V(y)$, $U(y)=\tilde{0}=V(x)$ and $U\subseteq\overline{V}$.

[cf. Srivastava-Ali [13]]

(6): (X,τ) is called $FT_2(vi)$ iff Δ is an IFCS in the product IFTS $(X\times X, \tau_{X\times X})$.

Theorem 3.7. Let (X,τ) be an IFTS. Then the following implications are valid:

$$FT_2(i) \Longrightarrow FT_2(v)$$

$$FT_2(i) \Longrightarrow FT_2(ii) \Longrightarrow FT_2(iii)$$

$$FT_2(iv) \Longleftrightarrow FT_2(v)$$

$$FT_2(i) \Longrightarrow FT_2(vi)$$

Proof. $(FT_2(i) \Rightarrow FT_2(v))$: This inclusion is obvious.

$(FT_2(i) \Rightarrow FT_2(ii))$: (a) Let $x(\alpha,\beta)$ and $y(\xi,\eta)$ be two distinct IFP's in X. Then there exist $U,V\in\tau$ such that $U(x)=\tilde{1}$, $V(y)=\tilde{1}$ and $U\cap V=\tilde{0}$. Then we have $\langle\alpha,\beta\rangle\le\tilde{1}=U(x)$, $\langle\xi,\eta\rangle\le\tilde{1}=V(y)$, and hence $x(\alpha,\beta)\subseteq U$ and $y(\xi,\eta)\subseteq V$ follow. Other properties can be shown similarly.

(b) Let $x(\beta)$ and $y(\eta)$ be two distinct VIFP's in X. Then there exist $U,V\in\tau$ such that $U(x)=\tilde{1}$, $V(y)=\tilde{1}$ and $U\cap V=\tilde{0}$. We have for $z\neq x$ $0\le\mu_U(z)$, $1\ge\gamma_U(z)$ and for $z=x$, $0\le\mu_U(x)$, $\beta\ge\gamma_U(x)=0$. Hence we obtain $x(\beta)=\langle z,0,1-x_{1-\beta}\rangle\subseteq U$ and, similarly, $y(\eta)=\langle z,0,1-y_{1-\eta}\rangle\subseteq V$.

$(FT_2(ii) \Rightarrow FT_2(iii))$: Let $x(\alpha,\beta)$ and $y(\xi,\eta)$ be two distinct IFP's in X. Then there exist $U,V\in\tau$ such that $x(\alpha,\beta)\subseteq U$, $y(\xi,\eta)\subseteq V$ and $U\cap V=\tilde{0}$. From the first inclusion we get $\langle\alpha,\beta\rangle\subseteq U(x)$, where $\alpha\neq 0$, $\beta\neq 1$, and from the other inclusion we get $\langle\xi,\eta\rangle\subseteq V(y)$, where $\xi\neq 0$, $\eta\neq 1$. Thus we have $U(x)\neq\tilde{0}$, $V(y)\neq\tilde{0}$. Other properties can be shown similarly.

$(FT_2(iv) \Rightarrow FT_2(v))$: Let $x,y\in X$, $x\neq y$. We consider the IFP's $x(1,0)$ and $y(1,0)$. Then there exist $U,V\in\tau$ such that $x(1,0)\subseteq U\subseteq\overline{y(1,0)}$, $y(1,0)\subseteq V\subseteq\overline{x(1,0)}$ and $U\subseteq\overline{V}$. From these inclusions we obtain $\langle 1,0\rangle\subseteq U(x)\subseteq\langle 1,0\rangle$, $\langle 1,0\rangle\subseteq V(y)\subseteq\langle 1,0\rangle$ and $\langle 0,1\rangle\subseteq U(y)\subseteq\langle 0,1\rangle$, $\langle 0,1\rangle\subseteq V(x)\subseteq\langle 0,1\rangle$. Thus $U(x)=\tilde{1}=V(y)$, $U(y)=\tilde{0}=V(x)$ and $U\subseteq\overline{V}$ follow directly.

$(FT_2(v) \Rightarrow FT_2(iv))$: (a) Let $x(\alpha,\beta)$ and $y(\xi,\eta)$ be two distinct IFP's in X. By our hypothesis, there exist $U,V\in\tau$ such that

$U(x)=\tilde{1}=V(y)$, $U(y)=\tilde{0}=V(x)$ and $U\subseteq\overline{V}$. Then $\alpha\leq 1=\mu_U(x)$, $\beta\geq 0=\gamma_U(x)$, $\xi\leq 1=\mu_V(y)$, $\eta\geq 0=\gamma_V(y)$ and $\mu_U(y)=0\leq\eta$, $\gamma_U(y)=1\geq\xi$, $\mu_V(x)=0\leq\beta$, $\gamma_V(x)=1\geq\alpha$, hence $x(\alpha,\beta)\subseteq U\subseteq\overline{y(\xi,\eta)}$, $y(\xi,\eta)\subseteq V\subseteq\overline{x(\alpha,\beta)}$ and $U\subseteq\overline{V}$ follow, as required.

(b) Let $x(\beta)$, $y(\eta)$ be two distinct VIFP's in X. Since $x\neq y$, there exist $U,U\in\tau$ such that $U(x)=\tilde{1}=V(y)$, $U(y)=\tilde{0}=V(x)$ and $U\subseteq\overline{V}$. We have for $z\neq x$, $0\leq\mu_U(z)$, $1\geq\gamma_U(z)$ and for $z=x$ $0\leq\mu_U(x)$, $\beta\geq\gamma_U(x)=0$. Hence we get $x(\beta)=\langle z,0,1-x_{1-\beta}\rangle\subseteq U$. For $z\neq y$, $\mu_U(z)\leq 1$, $\gamma_U(z)\geq 0$ and for $z=y$, $0=\mu_U(y)\leq\eta$, $\gamma_U(y)\geq 0$. Then $U\subseteq\langle z,1-y_{1-\eta},0\rangle=\overline{y(\eta)}$, $x(\beta)\subseteq U\subseteq\overline{y(\eta)}$ and, similarly, $y(\eta)\subseteq V\subseteq\overline{x(\beta)}$ follow.

(FT_2(i) $\Rightarrow$ FT_2(iv)): [Notice that FT_2(iv) $\Longleftrightarrow$ FT_2(v)] Since $U\cap V=\underset{\sim}{0}$ $\Longrightarrow$ $U\subseteq\overline{V}$, this implication is obvious.

(FT_2(i) $\Rightarrow$ FT_2(vi)): First of all, we may write down

$$\overline{\Delta}=(\bigcup\{\ (x,y)(\alpha,\beta)\ :(x,y)(\alpha,\beta)\leq\overline{\Delta}\})\cup(\bigcup\{\ (x,y)(\beta)\ :(x,y)(\beta)\leq\overline{\Delta}\}).$$

Now let $(x,y)(\alpha,\beta)\leq\overline{\Delta}$. But this means that $0<\alpha\leq\mu_{\overline{\Delta}}(x,y)=\gamma_{\Delta}(x,y)=1\Longrightarrow$ $x\neq y$. Hence, there exist $U,V\in\tau$ such that $U(x)=\tilde{1}$, $V(y)=\tilde{1}$ and $U\cap V=\underset{\sim}{0}$, from which we obtain $(x,y)(\alpha,\beta)\ \leq\ U\times V\ \leq\ \overline{\Delta}$. Similarly, if $(x,y)(\beta)\leq\overline{\Delta}$, then from $1>\beta\geq\gamma_{\overline{\Delta}}(x,y)=\mu_{\Delta}(x,y)=0$, we get $x\neq y$. Hence, there exist $U,V\in\tau$ such that $U(x)=\tilde{1}$, $V(y)=\tilde{1}$ and $U\cap V=\underset{\sim}{0}$, from which we again obtain $(x,y)(\beta)\ \leq\ U\times V\ \leq\ \overline{\Delta}$. Therefore $\overline{\Delta}$ is an IFOS in $X\times X$, in other words, Δ is an IFCS in $X\times X$ $\Longrightarrow$ (X,τ) is FT_2(vi). ■

Couterexample 3.8. Let $X=\{a,b\}$ and we consider IFS's U,V as follows:

$$U=\langle\ x,\ (0.4,\ 0.0),\ (0.1,\ 1.0)\ \rangle,$$
$$V=\langle\ x,\ (0.0,\ 0.3),\ (1.0,\ 0.2)\ \rangle.$$

The family $\tau=\{\underset{\sim}{0},\underset{\sim}{1},U,V,U\cup V\}$ is an IFT on X. Then (X,τ) is FT_2(iii), but not FT_2(ii). This is because for $\alpha=0.5$, $\beta=0.4$ and $\xi=0.5$, $\eta=0.3$, the property (a) of FT_2(ii) does not hold.

Counterexample 3.9. Let $X=\{a,b,c\}$ and define the IFS's A,B,C,D,E,F as follows:

$$A=\langle x, (1.0, 0.0, 0.1), (0.0, 1.0, 0.4) \rangle,$$
$$B=\langle x, (1.0, 0.4, 0.0), (0.0, 0.5, 1.0) \rangle,$$
$$C=\langle x, (0.0, 1.0, 0.3), (1.0, 0.0, 0.2) \rangle,$$
$$D=\langle x, (0.0, 0.3, 1.0), (1.0, 0.6, 0.0) \rangle,$$
$$E=\langle x, (0.3, 0.0, 1.0), (0.2, 1.0, 0.0) \rangle,$$
$$F=\langle x, (0.1, 1.0, 0.0), (0.5, 0.0, 1.0) \rangle.$$

Let τ denote the IFT on X generated by the subbase $\mathcal{S}=\{A,B,C,D,E,F\}$. Then (X,τ) is $FT_2(v)$, but not $FT_2(i)$. Indeed, if we consider the distinct elements a and b in X, and

$$U=A=\langle x, (1.0, 0.0, 0.1), (0.0, 1.0, 0.4) \rangle,$$
$$V=C=\langle x, (0.0, 1.0, 0.3), (1.0, 0.0, 0.2) \rangle,$$

then we have $U(a)=\tilde{1}=V(b)$, but the condition $U\cap V=\tilde{0}$ is not satisfied.

Theorem 3.10. Let (X,τ) be an IFTS. Then

$$FT_2(v) \Rightarrow FT_1(ii) ,$$
$$FT_2(iv) \Rightarrow FT_1(iii) .$$

Proof. Obvious. ■

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