A Note on RS-Compact Symmetric Topological Molecular Lattices

A. Haydar Eş and Doğan Çoker
Department of Mathematics - Department of Mathematics Education
Hacettepe University, Beytepe, 06532 -Ankara / TURKEY

Abstract: S-closed symmetric topological molecular lattices were first introduced by Wang [11]. In this paper, we discuss RS-compact symmetric topological molecular lattices.

Keywords: Symmetric topological molecular lattice, regular open (closed) set, S-closed space, RS-compact space, regular semiopen (semiclosed) set, extremally disconnected space.

1. Introduction

The concept of a S-closed space and RS-compact space has been investigated thoroughly in general topology [5, 6, 3], and has been generalized into the theory of topological lattices [1, 2, 7, 8, 9, 10, 11]. The aim of this paper is to continue the discussion of RS-compactness in symmetric topological molecular lattices.

A symmetric topological molecular lattice (briefly, symmetric TML) is a pair (L, η) where L is a fuzzy lattice, i.e., a completely distributive complete lattice with an order reversing involution " ′ " on it, and η is a co-topology on L, i.e., $\eta \subset L$, $0, 1 \in \eta$ and η is closed under the operations of finite unions and arbitrary intersections. (L, η) can also be written as (L(M), η), where M is the set consisting of all molecules e's. Here e e is called a molecule, if $e \neq 0$ and $e \leq A \vee B$ implies $e \leq A$ or $e \leq B$ for every pair A, B of elements of L. Members of η are called closed elements, members of $\delta = \eta' = \{P' : P \in \eta\}$ are called open elements.

An L-fuzzy topological space (briefly, L-fts) (L^X, δ) is a special symmetric TML, where $\eta = \delta'$, and so the concept of a symmetric TML is a generalization of the concept of an L-fts [11].

Definition 1.1. [11] $\forall\ A \in L$, $\overline{A} = \wedge\ \{P :\ A \leq P \in \eta\}$

$$\overset{0}{A} = \vee \{U : U \leq A \text{ and } U \in \delta\}$$

are the closure and interior of A, respectively.

Proposition 1.2. [11] $\forall A \in L$, $A'^{-\prime} = \overset{0}{A}$, $A'^{0\prime} = \overline{A}$

Definition 1.3. [11] Let (L, η) be a symmetric TML. An element A of L is called regular open, if $A = \overline{\overset{0}{A}}$; an element A of L is called regular closed, if $A = \overset{0}{\overline{A}}$.

Proposition 1.4. [11] A is regular open iff A′ is regular closed.

Definition 1.5. [11] Let (L, η) be a symmetric TML. An element A of L is semiopen, if there exists an open element V such that $V \leq A \leq \overline{V}$. A is semiclosed, if there exists a closed element Q such that $Q^0 \leq A \leq Q$.

Proposition 1.6. [11] Let (L, η) be a symmetric TML and A an element of L, then

(i) A is semiopen iff A′ is semi-closed.

(ii) If A is open, then A is semi-open, if A is regular open, then A is also semi-closed.

Proposition 1.7. [11] Let (L, η) be a symmetric TML and A an element of L, then the following conditions are equivalent to each other:

(i) (L, η) is extremally disconnected.

(ii) Closures of open elements are open.

(iii) Regular closed elements are open.

(iv) Regular open elements are closed.

Let L be a fuzzy lattice, $A \in L$ and $B = \{B_t\}_{t\in T} \subset L$. We say that B is a cover of A if $A \leq \bigvee_{t\in T} B_t$. We say that B is central, if the intersections of elements of finite subfamilies of B are non-zero [11].

Definition 1.8. [11] a) Let (L, η) be a symmetric TML. (L, η) is said to be S-closed, if every regular closed cover of the greatest element 1 has a finite subcover.

b) Let (L, η) be a symmetric TML. (L, η) is said to be H(i) if for every open cover $U = \{U_t\}_{t\in T}$ of 1, there exist $t_1, \ldots, t_n \in T$ such that $\{\overline{U}_{t_1}, \ldots, \overline{U}_{t_n}\}$ is a cover of 1.

Theorem 1.9. [11] Let (L, η) be an extremally disconnected symmetric TML, then (L, η) is S-closed iff it is H(i) (or almost compact).

Definition 1.10. [11] Let (L, η) be a symmetric TML η^* the co-topology generated by the closed base μ consisting of all regular closed elements of η, then (L, η^*) is called the semiregularization of (L, η).

Lemma 1.11. [11] Let (L, η) be a symmetric TML and (L, η^*) its semiregularization, A $\in$L. Then

(i) A is regular closed in (L, η) iff A is regular closed in (L, η^*).

(ii) A is regular open in (L, η) iff A is regular open in (L, η^*).

Theorem 1.12. [11] Let (L, η) be a symmetric TML and (L, η^*) its semiregularization, then (L, η) is S-closed iff (L, η^*) is S-closed.

Definition 1.13. [8] Let L_1 and L_2 be two fuzzy lattices. A mapping f: $L_1 \to L_2$ is said to be an order-homomorphism, if

(i) f is union-preserving,

(ii) f^{-1}: $L_2 \to L_1$ is involution preserving.

Definition 1.14. [8] Let (L_1,η_1) and (L_2,η_2) be symmetric TML's and f: $L_1 \to L_2$ an order-homomorphism. Then f is said to be an S-order-homomorphism, if for every regular closed element Q of (L_2,η_2), f^{-1}(Q) is a union of regular closed elements of (L_1,η_1).

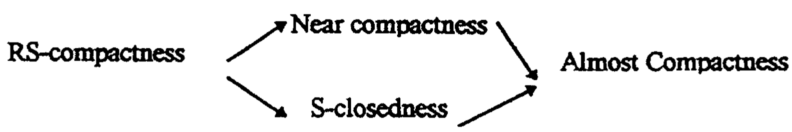
2. RS-Compact Symmetric Topological Molecular Lattices

Definition 2.1. Let (L, η) be a symmetric TML. An element A of L is regular semi-open, if there exists a regular open element U such that $U \le A \le \overline{U}$. A is regular semi-closed, if there exists a regular closed element B such that $\overset{0}{B} \le A \le B$.

Definition 2.2 a) Let (L, η) be a symmetric TML. (L, η) is said to be RS-compact, if every regular semi-open cover of the greatest element 1 has a finite subcover.

b) Let (L, η) be a symmetric TML. (L, η) is said to be nearly compact if every open cover U = $\{U_i\}_{i \in I}$ of 1, there exist i_1, …,i_n $\in$I such that $\{\overset{0}{\overline{U}}_{i_1},\ldots,\overset{0}{\overline{U}}_{i_n}\}$ is a cover of 1, or equivalent, every regular open cover of the greatest element 1 has a finite subcover.

It is clear that in TML's we have the following implications:



Theorem 2.3. Let (L, η) be an extremally disconnected symmetric TML. Then the following conditions are equivalent:

(i) (L, η) is RS-compact.

(ii) (L, η) is almost compact (or H(i)).

(iii) (L, η) is nearly compact.

(iv) (L, η) is S-closed.

Proof. Suppose that (L, η) is H(i) and $\mathcal{U}=\{U_i\}_{i\in I}$ is a regular semi-open cover of 1. Then there exists a regular open G_i such that $G_i \leq U_i \leq \overline{G}_i$, for each $i\in I$. Hence there exist $i_1,\ldots,i_n\in I$ such that $\{\overline{G}_{i_1},\ldots,\overline{G}_{i_n}\}$ covers 1. Since

$\overset{0}{\overline{U}}_i = \overset{0}{\overline{G}}_i = G_i = \overline{G}_i \leq U_i$, $\{U_{i_1},\ldots,U_{i_n}\}$ covers 1, hence (L,η) is RS-compact.

Theorem 2.4. Let (L,η) be a symmetric TML, then (L,η) is nearly compact iff every central family of regular closed elements has a non-zero intersection.

Proof. ⇒: Let $\mathcal{B}=\{B_i\}_{i\in I}$ be a central family of regular closed elements and $\bigwedge_{i\in I} B_i=0$. Then $\bigvee_{i\in I} B_i'=1$. It follows that there exist $i_1,\ldots,i_n \in I$ such that

$$B_{i_1}' \vee \ldots \vee B_{i_n}'=1.$$

Hence $\bigwedge_{m=1}^{n} B_{i_m}=0$, which is a contradiction.

⇐: Let $\mathcal{U}=\{U_i\}_{i\in I}$ be an open cover of 1 and $\bigvee_{i\in I}^{n} \overset{0}{\overline{U}}_i \neq 1$. Then $\{\overset{0}{\overline{U}}_i{}'\}_{i\in I}$ is a regular closed collection with $\bigwedge_{i=1}^{n} \overset{0}{\overline{U}}_i{}' \neq 0$. It follows that $\bigvee_{i\in I} \overset{0}{\overline{U}}_i \neq 1$. Hence $1= \bigvee_{i\in I} U_i \leq \bigvee_{i\in I} \overset{0}{\overline{U}}_i \neq 1$, which is a contradiction.

Theorem 2.5. (L,η) be a symmetric TML, then (L,η) is RS-compact iff for every regular semi-closed (or, regular semi-open) family $\mathcal{U}=\{U_i\}_{i\in I}$ such that $\bigwedge_{i\in I} U_i=0$, there exist finitely many $i_1,\ldots,i_n\in I$ with $\bigwedge_{m=1}^{n} U_{i_m}=0$.

Proof. ⇒: Let $U=\{U_i\}_{i\in I}$ be a regular semi-closed family and $\bigwedge_{i\in I} U_i=0$. Then $\{U_i'\}_{i\in I}$ is an regular semi-open cover of 1. From the hypothesis, there exist $i_1,\ldots,i_n\in I$ such that

$$U'_{i_1} \vee \ldots \vee U'_{i_n} = 1.$$

Now $\bigwedge_{m=1}^{n} U_{i_m}=0$ follows.

⇐: Similar to the above, and is omitted.

Theorem 2.6. Let (L,η) be a symmetric TML and (L,η^*) its semiregularization, then (L,η) is almost compact iff (L,η^*) is almost compact.

Proof. Similar to the proof Theorem 1.12 in [11].

Corollary 2.7. Let (L,η) be a symmetric TML and (L,η^*) its semiregularization, then (L,η) is RS-compact iff (L,η^*) is RS-compact.

In a symmetric TML, RS-compactness is not a "good extension" in the sense of Lowen [4]:

Example 2.8. Suppose that (L,η) is a fuzzy topological space $(X, W(U))$ which is the induced fuzzy space by W from the classical space (X, U). Then it is easy to see that (X, U) is RS-compact whenever $(X, w(U))$ is RS-compact. But the converse is not true. Let $X\neq\phi$ and $U=\{\phi,X\}$. Then (X,U) is RS-compact. On the other hand, $w(U)$ is the family of all constant functions defined on taking values in [0,1] [11]. Since $\wp=\{1 \frac{1}{n}: n=1,2,\ldots\}$ is a regular semi-open cover of 1 and it has no finite subcover of 1, $(X, W(U))$ is not RS-compact.

Definition 2.9. Let (L,η_1) and (L_2, η_2) be symmetric TML's and $f: L_1\to L_2$ an order homomorphism. Then f is said to be an RS-order-homomorphism, if for every regular semi-open element B of (L_2, η_2), $f^1(B)$ is a union of regular semi-open elements of (L_2,η_1).

Theorem 2.10. Let (L_1,η_1) be a symmetric TML and $f: (L_1,\eta_1)\to (L_2,\eta_2)$ be an RS-order-homomorphism which is surjective and (L_1,η_1) is RS-compact, then (L_2,η_2) is RS-compact.

Proof. Suppose $\{F_\alpha\}_{\alpha\in I}$ is a regular semi-open cover of 1_{L_2} then $\{f^1(F_\alpha)\}_{\alpha\in I}$ is cover of 1_{L_1}. Since f is an RS-order-homomorphism, $f^1(F_\alpha)$ is a union of regular semi-open elements of (L_1,η_1). Now the family $\bigcup_{\alpha\in I} B_\alpha$ is a regular semi-open cover of 1_{L_1}. Hence there exist finitely many $P_1,\ldots, P_n$ such that $\bigcup_{i=1}^{n} P_i = 1_{L_1}$ For each $1\le i\le n$, there exists an $\alpha_i \in I$ such that $P_i \le f^{1}(F_{\alpha_i})$. Then $f^{1}\left(\bigvee_{i=1}^{n} F_{\alpha_i}\right) = \bigvee_{i=1}^{n} f^{-1} F_{\alpha_i} \ge \bigvee_{i=1}^{n} P_i = 1_{L_1}$ and $\bigvee_{i=1}^{n} F_{\alpha_i} = 1_{L_2}$. Therefore (L_2,η_2) is RS-compact.

References

[1] S.L.Chen, NS-closedness in L-fuzzy topological spaces, Proc. IFES'91, Japan, 1991, Vol.1,27-32

[2] S.L.Chen, Several order-homomorphisms on L-fuzzy topological spaces, J.Shaanxi Normal University 16(1988) 15-19.

[3] W.C.Hong, On RS-compact spaces, J.Korean Math. Soc,17(1980) 39-43.

[4] R.Lowen, A comparision of different compactness notions in fuzzy topological spaces, J. Math. Anal. Appl. 64(1978) 446-454.

[5] T.Noiri, On RS-compact spaces, J. Korean. Math. Soc. 22(1985) 19-34.

[6] T.Noiri, On S-closed spaces, Atti Accad. Naz. Lince, Rend. Cl Sci. Fis Mat. Natur. (8) 64 (1978) 157-162.

[7] Guo-Jun Wang, Topological molecular lattices I, Science Bulletin, 29(1984) 19-23.

[8] Guo-Jun Wang, Order-homomorphisms on fuzzies, Fuzzy Sets and Systems, 12(1984) 281-288.

[9] Guo-Jun Wang, Pointwise topology on completely distributive lattices, Fuzzy Sets and Systems, 12(1984) 281-288.

[10] Guo-Jun Wang, Theory of topological molecular lattices, Fuzzy Sets and Systems, 47(1992) 351-376.

[11] Guo-Jun Wang, S-closed symmetric topological molecular lattices, The Journal of Fuzzy Math. 2(1994) 449- 461.