Fuzzy – Interval Distribution Numbers and the Interval Quasi – Probability Metrics

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Abstract. In this paper, we first introduce fuzzy – interval distribution numbers and their an extended add operation. And then on the basis of this, We study the relations between interval quasi – probability metrics and pseudo – metrics. Consequently another effective method is provied with further discussion of ordinary quasi – probability metric spaces.

Keywords Fuzzy – interval distribution numbers; extended addition; interval quasi – probability metrics; pseudo – metrics.

1 Introduction

Interval valued fuzzy sets were suggested at first by Gorzalczany M.B. [1] and Turksen I.B. [2]. They had been applied to the fields of engineering technique etc. Meng[3] studied interval valued fuzzy sets in detail and deeply, At the same time their decomposition theorems and representation theorems are established. Wang[4] defined ordinary interval valued fuzzy numbers on an closed bounded interval $[a,b] \subset R$, discussed some important properties and comparison problems of interval valued fuzzy sets. In this paper, we first define fuzzy – interval distribution numbers, and then give their an extended add operation. Second, by introducing the concepts of the interval quasi – probability metrics. Furthermore, we investagate the relations between them and ordinary pseudo – metrics. It will provide the applications of the theories of interval valued fuzzy sets with another effective method.

2 Fuzzy – interval distribution numbers

Throughout this paper, let $I = [0,1]$, R denote the set of all real numbers.

Write $[I] = \{[a,b] \mid a \leqslant b, a, b \in I\}$.

Definition 2.1. For any $[a_t, b_t] \in [I], t \in T$. Define

$$\bigwedge_{t\in T} a_t = inf\{a_t \mid t \in T\}; \qquad \bigvee_{t\in T} a_t = sup\{a_t \mid t \in T\};$$

$$\bigwedge_{t\in T} [a_t, b_t] = [\bigwedge_{t\in T} a_t, \bigwedge_{t\in T} b_t]; \qquad \bigvee_{t\in T} [a_t, b_t] = [\bigvee_{t\in T} a_t, \bigvee_{t\in T} b_t].$$

Especially, whenever $[a_1, b_1], [a_2, b_2] \in [I]$. We define

$[a_1, b_1] = [a_2, b_2]$ *iff* $a_1 = a_2$, $b_1 = b_2$;

$[a_1, b_1] \leqslant [a_2, b_2]$ *iff* $a_1 \leqslant a_2$, $b_1 \leqslant b_2$;

$[a_1, b_1] < [a_2, b_2]$ *iff* $[a_1, b_1] \leqslant [a_2, b_2]$ *and* $[a_1, b_1] \neq [a_2, b_2]$.

Obviously, $([I], \leqslant, \wedge, \vee)$ constitutes a complete lattice with a minimal element $\bar{0} = [0, 0]$ and a maximal element $\bar{1} = [1, 1]$.

Definition 2.2. Let a mapping $P: R \to [I]$, $x \to P(x) = [P^-(x), P^+(x)] \in [I]$. If the following conditions are satisfied.

(1) real functions P^- and P^+ are monotone nondecreasing;

(2) P^- and P^+ are left continous;

(3) $\lim_{x \to -\infty} P^-(x) = 0$, $\lim_{x \to +\infty} P^+(x) = 1$。

Then P is called a fuzzy – interval distribution number. If P satisfies $P(0) = \bar{0}$ besides above conditions. Then P is called a positive fuzzy – interval distribution number. Let $\triangle(H)$ denote the set of all fuzzy – interval distribution numbers. Let $\triangle(H_+)$ denote the set of all positive fuzzy – interval distribution numbers.

Whenever $P \in \triangle(H_+)$, obviously, if $x \leqslant 0$. Then we have $\bar{0} \leqslant P(x) \leqslant P(0) = \bar{0}$. i, e, $p(x) \equiv \bar{0}$. Therefore, we may regard positive fuzzy – interval distribution numbers as a mapping from $[0, \infty)$ to $[I]$.

In fact, fuzzy – interval distribution numbers are special interval valued fuzzy sets. P^- and P^+ are ordinary fuzzy sets.

Definition 2.3 Let $P, Q \in \triangle(H_+)$, define order "$\leqslant$"; $P \leqslant Q$ iff $Q(x) \leqslant P(x)$, for all $x \in [0, +\infty)$

Especially, $(\triangle(H_+), \leqslant)$ constitutes a partial ordered set.

$$\text{Letting } H(x) = \begin{cases} \bar{0} & x \leqslant 0 \\ \bar{1} & x > 0 \end{cases}$$

We easy kmow that $H \in \triangle(H_+)$ and H is a minimal element in $\triangle(H_+)$.

Definition 2.4. Let $P \in \triangle(H)$. For arbitrary $[\lambda_1, \lambda_2] \in [I]$. Let $P_{[\lambda_1, \lambda_2]} = \{x \in R \mid P^-(x) \geqslant \lambda_1, P^+(x) \geqslant \lambda_2\}$. Then $P_{[\lambda_1, \lambda_2]}$ is called a $[\lambda_1, \lambda_2]$ – level set of P.

By the definition of fuzzy – interval distribution numbers, we can prove that $P_{[\lambda_1, \lambda_2]}$ is a closed interval formed as $[a, +\infty)$ on R.

Definition 2.5. Let $P, Q \in \triangle(H)$. For all $z \in R$. we define

$(P \oplus Q)(z) = \bigvee_{z = x + y} [P^-(x) \wedge Q^-(y), P^+(x) \wedge Q^+(y)]$. Then $(P \oplus Q)$ is said to an extended add operation of fuzzy – interval distribution numbers.

Theorem 2.1. Let $P, Q \in \triangle(H)$. Then $(P \oplus Q) \in \triangle(H)$. In particularly, if $P, Q \in \triangle(H_+)$. Then $(P \oplus Q) \in \triangle(H_+)$.

Proof. By definition 2.5, it is easy to prove the monotonicity with respect to $(P \oplus Q)$. Now we proof its left continuity. For each $z_0 \in R$, we choose a sequence $z_n \uparrow z_0$. By definition 2.5 and the definition of supremum, for any $\varepsilon > 0$, there exists $x_0, y_0 \in R$, such that $z_0 = x_0 + y_0$ and $P^-(x_0) \wedge Q^-(y_0) \geqslant (P \oplus Q)^-(z_0) - \frac{\varepsilon}{2}$ where $(P \oplus Q)(z_0) = [(P \oplus Q)^-(z_0), (P \oplus Q)^+(z_0)]$

Since $\wedge$ is T-module continous. Thus for above $\varepsilon > 0$, as for P and Q, there exists a common $\delta > 0$ such that $(P^-(x_0) - \delta) \wedge (Q^-(y_0) - \delta) \geqslant P^-(x_0) \wedge Q^-(y_0) - \frac{\varepsilon}{2}$.

Further, letting sequences $x_n \uparrow x_0$, $y_n \uparrow y_0$ and $z_n = x_n + y_n$

Then from the left continuity of P^- and Q^-, for above $\delta > 0$, there exists a natural number N, whenever $n \geqslant N$, we obtain $P^-(x_n) \geqslant P^-(x_0) - \delta$ and $Q^-(y_n) \geqslant Q^-(y_0) - \delta$。 consequently,

$$(P \oplus Q)^-(z_n) \geqslant P^-(x_n) \wedge Q^-(y_n) \geqslant (P^-(x_0) - \delta) \wedge (Q^-(y_0) - \delta)$$

$$\geqslant P^-(x_0) \wedge Q^-(y_0) - \frac{\varepsilon}{2} \geqslant (P \oplus Q)^-(z_0) - \varepsilon.$$

i.e, $(P \oplus Q)^-$ is left continous. We can prove that $(P \oplus Q)^+$ is also left continous by the similar way.

Second, By $\wedge$ ia T-module continous, then for any $\varepsilon > 0$. We choose $0 < \delta \leqslant \varepsilon$. From the left continuity of P^- and Q^-, we know that there exists $x_0, y_0 \in R$, such that $P^-(x_0) \geqslant 1 - \delta$ and $Q^-(y_0) \geqslant 1 - \delta$.

By the monotonicity of $(P \oplus Q)^-$, we can obtain that $(P \oplus Q)^-(z) \geqslant P^-(x_0) \wedge Q^-(y_0) \geqslant 1 - \delta \geqslant 1 - \varepsilon$ whenever $z > x_0 + y_0$. Thus $\lim_{z \to +\infty} (P \oplus Q)^-(z) = 1$.

Similarly, $\lim_{z \to +\infty} (P \oplus Q)^+(z) = 1$ i.e., $\lim_{z \to +\infty} (P \oplus Q)(z) = \bar{1}$. By the similar method, we get $\lim_{z \to -\infty} (P \oplus Q)(z) = \bar{0}$. The proof is omitted.

Finally, if $P, Q \in \triangle(H_+)$. From $P(0) = \bar{0}$ and $Q(0) = \bar{0}$, we show that $(P \oplus Q)^-(0) = (P \oplus Q)^+(0) = 0$. Therefore $(P \oplus Q) \in \triangle(H_+)$.

Theorem 2.2 Let $P, Q \in \triangle(H)$. Then for arbitrary $[\lambda_1, \lambda_2] \in [I]$,

$$(P \oplus Q)_{[\lambda_1, \lambda_2]} = P_{[\lambda_1, \lambda_2]} + Q_{[\lambda_1, \lambda_2]}$$

Proof. By definition 2.4 and definition 2.5, we can proof it easily. So omit it.

3. Interval quasi-probability metrics.

Definition 3.1. Let X be an ordinary set, mapping $D: X \times X \to \triangle(H_+)$. suppose D fulfills

(1) $D(x, x) = H$, (2) $D(x, y) = D(y, x)$,

(3) $D(x, y) \leqslant D(x, z) \oplus D(z, y)$, $\forall x, y, z \in X$

Then D is called an interval quasi-probability metric on X. and triple $(X, D, \oplus)$ is called an interval quasi-probability metric space

Definition 3.2 Let mapping $\rho: X \times X \to [0, +\infty)$. Assume ρ satisfies

(1) $\rho(x, x) = 0$ (2) $\rho(x, y) = \rho(y, x)$

(3) $\rho(x, y) \leqslant \rho(x, z) + \rho(z, y)$, $\forall x, y, z \in X$

Then ρ is said to a pseudo-metric on X, and (X, ρ) is said to a pseudo-metric space.

Theorem 3.1 Let $(X, D, \oplus)$ be an interval quasi-probability metric space. For any $[\lambda_1, \lambda_2] \in [I]$. Let $d_{[\lambda_1,\lambda_2]}(x, y) = inf\{z \in R \mid D(x, y)(z) \geqslant [\lambda_1, \lambda_2]\}$. Then $(X, d_{[\lambda_1,\lambda_2]})$ is a pseudo-metric space.

Proof. For each $[\lambda_1, \lambda_2] \in [I]$, From $D(x, x) = H$, we have $(D(x, x))_{[\lambda_1,\lambda_2]} = H_{[\lambda_1,\lambda_2]} = [0, +\infty)$.

Thus $d_{[\lambda_1,\lambda_2]}(x, x) = 0$ and $d_{[\lambda_1,\lambda_2]}(x, y) = d_{[\lambda_1,\lambda_2]}(y, x)$ is obvious.

Further, by definition 2.3 and $D(x, y) \leqslant D(x, z) \oplus D(z, y)$,
we view $D(x, y)(z') \geqslant (D(x, z) \oplus D(z, y))(z')$, for any $z' \in R$.

Hence $(D(x, y))_{[\lambda_1,\lambda_2]} \supset (D(x, z) \oplus D(z, y))_{[\lambda_1,\lambda_2]}$ Taking infimum at two sides at the same time. We obtain $d_{[\lambda_1,\lambda_2]}(x, y) \leqslant d_{[\lambda_1,\lambda_2]}(x, z) + d_{[\lambda_1,\lambda_2]}(z, y)$.

Therefore $(X, d_{[\lambda_1,\lambda_2]})$ is a pseudo-metric space.

Theorem 3.2 Let $(X, d_{[\lambda_1,\lambda_2]})$ be a pseudo-metric space. If d is monotone increasing with respect to $[\lambda_1, \lambda_2] \in [I]$. Let $D(x, y)(z) = \bigvee_{[\lambda_1,\lambda_2] \in [I]} \{[\lambda_1, \lambda_2] \mid d_{[\lambda_1,\lambda_2]}(x, y) \leqslant z\}$.

Then D is an interval quasi-probability metric.

Proof. Simulating the proof of theorem 2.1, we can prove $D(x, y) \in \Delta(H_+)$. so omit it
Obviously, $D(x, x) = H$ and $D(x, y) = D(y, x)$ for any $x, y \in X$.

Applying definition 2.4 and the theorem of nested sets with respect to interval valued fuzzy sets in [3], for any $[\lambda_1, \lambda_2] \in [I]$. We have

$$(D(x, y))_{[\lambda_1,\lambda_2]} = \bigcup_{[\lambda',\lambda''] \geqslant [\lambda_1,\lambda_2]} [d_{[\lambda',\lambda'']}(x, y), +\infty) = [\inf_{[\lambda',\lambda''] \geqslant [\lambda_1,\lambda_2]} d_{[\lambda',\lambda'']}(x, y), +\infty)$$

since $d_{[\lambda',\lambda'']}(x, y) \leqslant d_{[\lambda',\lambda'']}(x, z) + d_{[\lambda',\lambda'']}(z, y)$, $\forall [\lambda', \lambda''] \in [I]$

Hence $(D(x, z) \oplus D(z, y))_{[\lambda',\lambda'']} = (D(x, z))_{[\lambda',\lambda'']} + (D(z, y))_{[\lambda',\lambda'']} \subset (D(x, y))_{[\lambda',\lambda'']}$

By the decomposition theorem of interval valued fuzzy sets in [3], we have
$(D(x, z) \oplus D(z, y))(z') \leqslant D(x, y)(z')$, for each $z' \in R$. From definition 2.3,
We show that $D(x, y) \leqslant D(x, z) \oplus D(z, y)$. Hence D is an interval quasi-probability metric.

References

〔1〕Gorzafczany B., Approximate Inference with interval - valued fuzzy sets - an outline, in: proc. Polish symp. on Interval and Fuzzy math, poznan, 1983.89 - 95.

〔2〕Turksen B.. Interval - valued fuzzy sets based on normal forms. Fuzzy sets and systems. 1986, 20: 191 - 210.

〔3〕Meng Guang wu. The basic theories of interval - valued fuzzy sets. Mathematic Applicate 1993, vol. 6. No. 2: 212 - 217.

〔4〕Wang Guijun and Li Xiaoping. Correlation and information energy of interval valued fuzzy numbers. BUSEFAL. 64 (1995). 3 - 8.

〔5〕Grattan - Guiness, I. Fuzzy membership mapped onto interval and many valued quantities, Z, Math. Logik Grundlag. Math. 22(1975), 149 - 160.

〔6〕Jahn, K. U. Interval - wertige mengen. Math. Nachr., 68(1975), 115 - 132.

〔7〕Zhang Wenxiu. Basis of Fuzzy Mathematics (Xi'an Jiao tong univ, press, 1991).