Some Properties of Fuzzy Set-valued Martingales

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Abstract:

In this paper, we discuss some properties of fuzzy set-valued martingales, and give a series of Doob stopping theorems of fuzzy set-valued martingales, fuzzy set-valued submartingales and fuzzy set-valued supermartingales

Keywords: fuzzy set-valued stochastic process, fuzzy set- valued martingale(submartingale,supermartingale), Doob stopping Theorem.

1. Introduction

It is well known that set-valued functions and set-valued random variables have been used repeatedly in economics [4], [9], [10]. The conditional expections of set-valued random variable and set-valued martingales have been studied by Hiai.F, Papageorgion.N.S, Zhang Wenxiu, Nie Zhan-kan and Gao Yong e.t. The discussions of fuzzy set-valued martingles have been oringnated ([1], [6]). This paper's purpose is to discuss further fuzzy set-valued martingales and fuzzy set-valued stohastic processes, and to get some more deep results.

In section 2, elemental notions of fuzzy set-valued stochastic process and fuzzy set-valued martingales are given.

In section 3, we discuss Doob stopping theorems of set- valued submartingales, set-valued martingales and set-valued supermartingales.

In section 4, we will give Doob stopping theorems of fuzzy set-valued submartingales, fuzzy set-valued martingales, and fuzzy set-valued supermartingales.

2. Notions of Fuzzy Set-valued Stochastic Processes and Fuzzy Set-valued Martingales

Let X be a n-dimension Euclidean space in this paper, $(\Omega, \mathcal{F}, P)$ be a complete probability measure space, $\{\mathcal{F}_t\}_{t \in R_+}$ be a family of monotone increasing sub-σ-fields of $\mathcal{F}$, and $\mathcal{F}_\infty = \bigvee_{t \in R_+} \mathcal{F}_t$ in this paper, where R_+ is the set of all nonnegative real numbers.

Let $\tilde{F}_0(X)$ be the family of all fuzzy sets $\tilde{A}: X \to [0, 1]$ with properties:

(1) $\tilde{A}$ is upper semicontinuous,

(2) $\tilde{A}$ is fuzzy convex,

(3) $\tilde{A}_\alpha$ is compact for every $\alpha \in (0,1]$.

where $\tilde{A}_\alpha = \{x \in X: \tilde{A}(x) \geqslant \alpha\}$ is the α-level set of $\tilde{A}$.

If $\tilde{A}, \tilde{B} \in \tilde{\mathcal{F}}_0(X)$, difine the distance between $\tilde{A}$ and $\tilde{B}$ by

$$d(\tilde{A}, \tilde{B}) = \sup_{\alpha > 0} h(\tilde{A}_\alpha, \tilde{B}_\alpha)$$

where h denotes the Hausdorff distance.

2.1 Theorem

$(\tilde{\mathcal{F}}_0(X), d)$ is a complete metric space.

Proof: See theorem 4.1 and 4.2 in [8].

A linear structure is defined in $\tilde{\mathcal{F}}_0(X)$ by

$$(\tilde{A} + \tilde{B})(x) = \sup\{\alpha \in [0,1]: x \in (\tilde{A}_\alpha + \tilde{B}_\alpha)\}$$

$(\lambda\widetilde{A})(x)=\widetilde{A}(\lambda^{-1}x)$, if $\lambda\neq 0$

$(\lambda\widetilde{A})(x)=0$, if $\lambda=0, x\neq 0$

$(\lambda\widetilde{A})(x)=\sup_{y\in X}\widetilde{A}(y)$ if $\lambda=0, x=0$.

for $\widetilde{A},\widetilde{B}\in\widetilde{\mathcal{F}}_0(X), \lambda\in R$. It is easy to prove that $(\widetilde{A}+\widetilde{B})_\alpha=\widetilde{A}_\alpha+\widetilde{B}_\alpha$, and $(\lambda\widetilde{A})_\alpha=\lambda\widetilde{A}_\alpha$ for every $\alpha\in[0,1]$.

2.2 Definition

Let $\widetilde{F}:(\Omega,\mathcal{F})\to\widetilde{\mathcal{F}}_0$ be a mapping from $(\Omega,\mathcal{F})$ to $\widetilde{\mathcal{F}}_0(X)$.

1. $\widetilde{F}$ is called a fuzzy set-valued random variable if

$$\{\omega:\widetilde{F}^{-1}(C)(\omega)\in B\}=\{\omega:\sup_{x\in C}\widetilde{F}(\omega)(x)\in B\}\in\mathcal{F}$$

for any closed subset C of X and Borel's subset B of $[0,1]$, i.e. $B\in\mathcal{B}([0,1])$.

2. $\widetilde{F}$ is called $\mathcal{F}$-level measurable if $\widetilde{F}_\alpha$ defined by $(\widetilde{F})_\alpha(\omega)=(\widetilde{F}(\omega))_\alpha$ for every $\omega\in\Omega$ is a random set for every $\alpha\in(0,1]$.

The following two properties are equavalent:

1. $\widetilde{F}$ is a fuzzy set-valued random variable.
2. $\widetilde{F}$ is $\mathcal{F}$-level measurable .

Proof: See theorem 1.5.1 in [1].

2.3 Definition

Let T be a subset of R. If for each $t\in T$, there is a fuzzy set-valued random variable $\widetilde{F}_t$, then $\{\widetilde{F}_t\}_{t\in T}$ is called a fuzzy set-valued stochastic process

2.4 Defintion

Let $\widetilde{F}:(\Omega,\mathcal{F},P)\to\widetilde{\mathcal{F}}_0(X)$ be a fuzzy set-valued random variable, $\widetilde{F}$

is called to be integrable bounded, if there exists a nonnegative integrable function σ_α for every $\alpha \in (0,1]$ such that $\|x\| < \sigma_\alpha(\omega)$ for every $x \in \widetilde{F}_\alpha(\omega), \omega \in \Omega$.

Define:

$$(\int_\Omega \widetilde{F} dP)(x) = \bigvee_{\alpha \in [0,1]} (\alpha \wedge I_{\int_\Omega \widetilde{F}_\alpha dP}(x))$$

where $\int_\Omega \widetilde{F}_0 dP = X$.

2.5 Theorem

Let $\widetilde{F}: (\Omega, \mathcal{A}, P) \to \widetilde{\mathcal{F}}_0(X)$ be an integrable bounded fuzzy set-valued random variable, then $(\int_\Omega \widetilde{F} dP)_\alpha = \int_\Omega \widetilde{F}_\alpha dP$ and $\int_\Omega \widetilde{F} dP \neq \Phi$

Proof: See theorem 6.5.3 in [1].

2.6 Theorem

Let $\widetilde{F}: (\Omega, \mathcal{F}, P) \to \widetilde{\mathcal{F}}_0(X)$ be an integrable bounded fuzzy set-valued random variable, then for every sub-field $\mathcal{F}_1$ of $\mathcal{F}$, there exists an unique $\mathcal{F}_1$-measurable fuzzy set-valued variable $\widetilde{G}$ such that

$$\int_A \widetilde{G} dP = \int_A \widetilde{F} dP \quad \text{for every } A \in \mathcal{F}_1.$$

Proof: See theorem 6.5.4 in [1].

2.7 Definition

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{F}_1$ be a sub-σ-field of $\mathcal{F}$, $\widetilde{F}: (\Omega, \mathcal{F}, P) \to \widetilde{\mathcal{F}}_0(X)$ be an integrable bouned fuzzy set-valued variable. $\widetilde{G}: (\Omega, \mathcal{F}, P) \to \widetilde{\mathcal{F}}_0(X)$ is called a conditional expectation of $\widetilde{F}$ with respect to the sub-σ-field $\mathcal{F}_1$ of $\mathcal{F}$, and is denoted by $\mathcal{E}[\widetilde{F}/\mathcal{F}_1]$, if $\widetilde{G}$ is a $\mathcal{F}_1$- measurable, integrable bounded fuzzy set-valued random variable satisfying the following condition:

$\int_A \tilde{G}\, dP = \int_A \tilde{F}\, dP$ for every $A \in \mathcal{F}_1$.

Conditional expectation of $\tilde{F}$ is a.e. unique and $(\mathcal{E}[\tilde{F}/\mathcal{F}_1](\omega))_\alpha$ $= \mathcal{E}[\tilde{F}_\alpha/\mathcal{F}_1](\omega)$ a.e. by Theorem 2.5 and 2.6.

2.8 Definition

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(\mathcal{F}_t)_{t \in R_+}$ be a family of monotone increasing sub-σ-fields of $\mathcal{F}$. A fuzzy set-valued stochastic process $(\tilde{F}_t)_{t \in R_+}$ is said to be $(\mathcal{F}_t)_{t \in R_+}$ - adapted, if $\tilde{F}_t$ is a $(\mathcal{F}_t)_{t \in R_+}$ - measurable fuzzy set-valued random variable. A $(\mathcal{F}_t)$ - adapted integrable bounded fuzzy set-valued stochastic process $(\tilde{F}_t)_{t \in R_+}$ is called a fuzzy set-valued martingale (resp. super-martingale, submartingale), if $\mathcal{E}[\tilde{F}_t/\mathcal{F}_s] = \tilde{F}_s$ (resp. $\subset \tilde{F}_s$, $\supset \tilde{F}_s$) a.e. for any $t, s \in R_+$, and $s < t$.

3. The Doob Stopping Theorems of Set-valued Martingales

3.1 Definition

A trajectory $F.(\omega)$ of a set-valued stochastic process $(F_t)_{t \in R_+}$ is said to be right continuous at t_0 with respect to h if $\lim_{\substack{t \to t_o \\ t > t_o}} h((F_t(\omega), F_{t_o}(\omega)) = 0$. A trajectory $F.(\omega)$ of a set-valued stochastic process $(F_t)_{t \in R_+}$ is said to be right continuous with respect to h if $\lim_{\substack{t \to s \\ t > s}} h((F_t(\omega), F_s(\omega)) = 0$ for every $s \in R_+$.

3.2 Theorem

$(F_t)_{t \in \bar{R}_+} \subset \mathcal{L}_c^1[\Omega, X]$ is a $(\mathcal{F}_t)_{t \in \bar{R}_+}$ - adapted set- valued super maringale(resp. set-valued martingale, set-valued submartingale) if and only if $(\sigma(x^*,F_t))_{t\in R_+}$ is a $(\mathcal{F}_t)_{t\in R_+}$-adapted supermartingale (resp. martingale, submartingale) for every $x^*\in X^*$. Where $\sigma(x^*, F_t(\omega))$ is a support function of $F_t(\omega)$, X^* is the dual space of X.

Proof: See theorem 4.1.5 and 4.1.6 in [3].

3.3 Theorem

Let $(\mathcal{F}_t)_{t\in\bar{R}_+}$ be right continuous, $(F_t)_{t\in\bar{R}_+}\subset\mathcal{L}_c^1[\Omega,X]$ be a $(\mathcal{F}_t)$-adapted set-valued supermartingale, its almost all trajectoriesbe right continuous with respect to h, and S,T be two stopping times. Then $\mathcal{E}[F_T/\mathcal{F}_S]\subset F_S$ a.e.

Proof: Since $(\sigma(x^*,F_t))_{t\in R_+}$ is a $(\mathcal{F}_t)_{t\in\bar{R}_+}$- adapted supermartingale for every $x^*\in X^*$ by Theorem 3.2. And a sequnce (A_n) of closed convex subsets in reflecxive space converged to A with respect to h, then (A_n) weak converge to A, i.e. $\sigma(x^*, A_n)$ converge to $\sigma(x^*,A)$ for every $x^*\in X^*$. Thus almost all trajectories of $(\sigma(x^*, F_t))_{t\in\bar{R}_+}$ is right continuous for every $x^*\in X^*$, and $\sigma(x^*,\mathcal{E}[F_T/\mathcal{F}_S])=E[\sigma(x^*,F_T)/\mathcal{F}_S]\leqslant\sigma(x^*,F_S)$ a. e. for any two stopping times $S,T,S\leqslant T$ and $x^*\in X^*$. Therefore $\mathcal{E}[F_T/\mathcal{F}_S]\subset F_S$ a.e.

Similarly, we have the following theorem:

3.4 Theorem

Let $(\mathcal{F}_t)_{t\in\bar{R}_+}$ be right continuous, $(F_t)_{t\in\bar{R}_+}\subset\mathcal{L}_c^1[\Omega,X]$ be a $(\mathcal{F}_t)_{t\in\bar{R}_+}$ - adapted set-valued submartingale (resp. set-valued martingale), its almost all trajectories be right continuous with respect to h. Then $\mathcal{E}[F_T/\mathcal{F}_S]\supset F_S$ (resp. $=F_S$) a.e. for any two stopping times T,S, and $T\geqslant S$.

Strong forms of Doob stopping theorems are discussed as follows:

3.5 Theorem

Let $(\mathcal{F}_t)_{t\in\bar{R}_+}$ be right continuous, $(F_t)_{t\in\bar{R}_+}\subset\mathcal{L}_c^1[\Omega, X]$ be a set - valued supermartingale,(resp. set- valued martingale, submartin -gale) and its almost all trajectories be right continuous with respect to h. If S,T are two stopping times, then

$$E[F_T/\mathcal{F}_S]\subset F_{T\wedge S} \quad (\text{resp. } =F_{T\wedge S},\ \supset F_{T\wedge S}) \text{ a.e.}$$

Proof: Since $(F_t)_{t\in\bar{R}_+}$ is a $(\mathcal{F}_t)_{t\in\bar{R}_+}$-adapted closed convex set-valued supermartingale (resp. set-valued martingale, submartingale) and its almost all trajectories are right continuous with respect to h, then $(\sigma(x^*, F_t))_{t\in\bar{R}_+}$ is a $(\mathcal{F}_t)_{t\in\bar{R}_+}$ - adapted supermartingale (resp. martingale, submartingale) and its almost all trajectories are right continuous for each $x^*\in X^*=R^n$. Therefore $\sigma(x^*, E[F_T/\mathcal{F}_S])=E[\sigma(x^*,F_T)/\mathcal{F}_S]\leqslant\sigma(x^*,F_{T\wedge S})$ (resp. $=\sigma(x^*,F_{T\wedge S})$, $\geqslant\sigma(x^*,F_{T\wedge S})$) a. e. for any $(\mathcal{F}_t)_{t\in\bar{R}_+}$ stopping times S,T. Thus $E[F_T/\mathcal{F}_S]\subset F_{T\wedge S}$ (resp. $=F_{T\wedge S}$, $\supset F_{T\wedge S}$) a.e. for any $(\mathcal{F}_t)_{t\in\bar{R}_+}$ stopping times T,S.

3.6 Theorem

Let $(\mathcal{F}_t)_{t\in\bar{R}_+}$ be right continuous, $(F_t)_{t\in\bar{R}_+}\subset\mathcal{L}_c^1[\Omega, X]$ be a $(\mathcal{F}_t)_{t\in\bar{R}_+}$-adapted set- valued supermartingale, (resp. set-valued martingale) and its almost all trajectories be right continuous with respect to h. If S,T are two large stopping times, then

$$E[F_T/\mathcal{F}_{S+}]\subset F_{T\wedge S} \quad (\text{resp. } =F_{T\wedge S}) \text{ a.e.}$$

Proof: Since $(\sigma(x^*,F_t))_{t\in\bar{R}_+}$ is a $(\mathcal{F}_t)_{t\in\bar{R}_+}$-adapted supermartingale for every $x^*\in X^*$ and its all trajectories are right continuous. Then $\sigma(x^*,E[F_T/\mathcal{F}_{S+}])=E[\sigma(x^*,F_T)/\mathcal{F}_{S+}]\leqslant\sigma(x^*,F_{S\wedge T})$ (resp. $=\sigma(x^*,F_{T\wedge S})$) a. e. for any two large stopping times T,S by classical results. Thus

$$E[F_T/\mathcal{F}_{S+}]\subset F_{T\wedge S} \quad (\text{resp. } =F_{T\wedge S}) \text{ a.e.}$$

The following theorem is the predictable form of Doob stopping theorem.

3.7 Theorem

Let $(\mathcal{F}_t)_{t\in\bar{R}_+}$ be right continuous, $(F_t)_{t\in\bar{R}_+}\subset\mathcal{L}_c^1[\Omega, X]$ be $(\mathcal{F}_t)_{t\in\bar{R}_+}$-adapted set- valued supermartingale (resp. set- valued martingale), and its almost all trajectories be right continuous with respect to h. Let $F_{0-}=\mathcal{E}[F_0/\mathcal{F}_{0-}]$, then

$$\mathcal{E}[F_S/\mathcal{F}_{T-}]\subset F_{T-} \text{ (resp.} =F_{T-}) \text{ a.e.}$$

for any predictable time T and stopping time $S\geqslant T$, where $\mathcal{F}_{0-}$ is a sub-σ-field of $\mathcal{F}_0$.

Proof: Since $(\sigma(x^*,F_t))_{t\in\bar{R}_+}$ is a $(\mathcal{F}_t)_{t\in\bar{R}_+}$ supermartingale for every $x^*\in X^*$ by Theorem 3.2, and almost all trajectories of $(\sigma(x^*,F_t))_{t\in\bar{R}_+}$ are right continuous by the proof of Theorem 3.3, and $\sigma(x^*,F_{0-})=\sigma(x^*,\mathcal{E}[F_0/\mathcal{F}_{0-}]=E[\sigma(x^*,F_0)/\mathcal{F}_{0-}]$ for every $x^*\in X^*$. Then $\sigma(x^*,\mathcal{E}[F_S/\mathcal{F}_{T-}])=E[\sigma(x^*,F_S)/\mathcal{F}_{T-}]\leqslant\sigma(x^*,F_{T-})$ (resp. $=\sigma(x^*,F_{T-})$) a.e. for every $x^*\in X^*$, predictable time T, and stopping time $S\geqslant T$ by classical results. Thus

$$\mathcal{E}[F_S/\mathcal{F}_{T-}]\subset F_{T-} \text{ (resp. } =F_{T-}) \text{ a.e.}$$

4. The Doob stopping Theorems of Fuzzy Set-valued Martingales

4.1 Theorem

Let $(\mathcal{F}_t)_{t\in\bar{R}_+}$ be right continuous, $(\tilde{F}_t)_{t\in\bar{R}_+}$ be a $(\mathcal{F}_t)_{t\in\bar{R}_+}$-adapted fuzzy set-valued supermartingale, and its almost all trajectories be right continuous with respect to d. Then

$$\mathcal{E}[\tilde{F}_T/\mathcal{F}_S]\subset\tilde{F}_S \text{ a.e.}$$

for any two stopping times T, S, and $T\geqslant S$.

Proof: Since $(\tilde{F}_t)_{t\in\bar{R}_+}$ is a $(\mathcal{F}_t)_{t\in\bar{R}_+}$- adapted fuzzy set-valued supermartingale and its almost all trajectories are right continuous with respect to d, then $((\tilde{F}_t)_\alpha)_{t\in\bar{R}_+}$ is a $(\mathcal{F}_t)_{t\in\bar{R}_+}$

set -valued supermartingale, its almost all trajectories are right continuous with respect to h for each $\alpha\in(0,1]$ by $\mathcal{E}[\widetilde{F}_t/\mathcal{F}_s])_\alpha=\mathcal{E}[(\widetilde{F}_t)_\alpha/\mathcal{F}_s]$ and $d(\widetilde{F}_t(\omega),\widetilde{F}_{t_0}(\omega))=\sup_{\alpha\in(0,1]} h((\widetilde{F}_t)_\alpha(\omega)),(\widetilde{F}_{t_0})_\alpha(\omega))$. Therefore, $\mathcal{E}[(F_T)_\alpha/\mathcal{F}_S]\subset(\widetilde{F}_S)_\alpha$ a.e. for any two stopping times T,S and $T\geqslant S$ by Theorem 3.3.

On the other hand, using the basic theorems of fuzzy set, $(\mathcal{E}[\widetilde{F}_T/\mathcal{F}_S](\omega))(x)=\bigvee_{\alpha\in R_0}\alpha\wedge I_{(\mathcal{E}[\widetilde{F}_T/\mathcal{F}_S](\omega))_\alpha}(x)$, where R_0 is the set of all rational numbers in $[0,1]$.

Therefore we have $\mathcal{E}[\widetilde{F}_T/\mathcal{F}_S]\subset\widetilde{F}_S$ a.e. for any $(\mathcal{F}_t)_{t\in\bar{R}_+}$ stopping times T,S with $T\geqslant S$.

Similarly, we have the following theorem.

4.2 Theorem

Let $(\mathcal{F}_t)_{t\in\bar{R}_+}$ be right continuous, $(\widetilde{F}_t)_{t\in\bar{R}_+}$ be a $(\mathcal{F}_t)_{t\in\bar{R}_+}$ fuzzy set-valued submartingale (resp. fuzzy set-valued martingale) and its almost all trajectories be right continuous with respect to d. Then

$$\mathcal{E}[\widetilde{F}_T/\mathcal{F}_S]\supset\widetilde{F}_S \quad (\text{resp. } =\widetilde{F}_S) \text{ a.e.}$$

for any two stopping times T,S, and $T\geqslant S$.

By basic theorems of fuzzy set and Theorem 3.5 to Theorem 3.7, correspondingly, the following strong forms and preditable form of Doob stopping theorems of fuzzy set-valued martingale(submartin-gale, supermartingale) can be obtained.

4.3 Theorem

Let $(\mathcal{F}_t)_{t\in\bar{R}_+}$ be right continuous, $(\widetilde{F}_t)_{t\in\bar{R}_+}$ be a fuzzy set -valued supermartingale (resp. fuzzy set-valued martingale, fuzzy set -valued submartingale) and its almost all trajectories be right continuous with respect to d. Then

$$\mathcal{E}[\widetilde{F}_T/\mathcal{F}_S] \subset \widetilde{F}_{T\wedge S} \text{ (resp. } =\widetilde{F}_{T\wedge S}, \supset \widetilde{F}_{T\wedge S}) \text{ a.e.}$$

for any two stopping times T,S.

4.4 Theorem

Let $(\mathcal{F}_t)_{t\in\bar{R}_+}$ be right continuous, $(\widetilde{F}_t)_{t\in\bar{R}_+}$ be a $(\mathcal{F}_t)_{t\in\bar{R}_+}$ fuzzy set-valued supermartingale (resp. fuzzy set-valued martingale, fuzzy set-valued submartingale) and its almost all trajectories be right continuous with respect to d. If S,T are two large stopping times, then

$$E[\widetilde{F}_T/\mathcal{F}_{S+}] \subset \widetilde{F}_{T\wedge S} \text{ (resp. } =\widetilde{F}_{T\wedge S}, \supset \widetilde{F}_{T\wedge S}) \text{ a.e.}$$

4.5. Theorem

Let $(\mathcal{F}_t)_{t\in\bar{R}_+}$ be right continuous, $(\widetilde{F}_t)_{t\in\bar{R}_+}$ be a $(\mathcal{F}_t)_{t\in\bar{R}_+}$ fuzzy set-valued supermartingale (resp. fuzzy set-valued martingale) and its almost all trajectories be right continuous with respect to d. Let $\widetilde{F}_{0-}=\mathcal{E}[\widetilde{F}_0/\mathcal{F}_{0-}]$, then

$$\mathcal{E}[\widetilde{F}_U/\mathcal{F}_{T-}] \subset \widetilde{F}_{T-} \text{ (resp. } =\widetilde{F}_{T-}) \text{ a.e.}$$

for any predictable time T and stopping time $U \geqslant T$.

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