FURTHER DISCUSSIONS ON GENERALIZED FUZZY INTEGRALS ON FUZZY SETS (Ⅱ): Generalized fuzzy intergrals of fuzzy-valued functions on fuzzy sets

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Abstract

This paper is a continuous work of [8], basing on Wu's generalized fuzzy integrals of point-valued function on fuzzy sets [6], the theory of generalized fuzzy integrals of fuzzy-valued functions on fuzzy sets has been established. These are good extensions of results in [6, 7, 8].

Keywords: Fuzzy measure on fuzzy sets, Generalized fuzzy integrals on fuzzy sets, Fuzzy-valued function.

1. Introduction

Since Sugeno[4] brought out the concepts of fuzzy measures and fuzzy integrals, the theory has been made much deeper by many others. Especially, the generalized fuzzy integral on fuzzy sets (in short, (G) fuzzy integral on fuzzy sts) introduced by Wu[6] is a much wider and good one. Meanwhile, the theory of fuzzy-valued functions (a special kind of fuzzy random variable) is rapidly developed. In this paper, we will show (G) fuzzy integrals of fuzzy-valued function on fuzzy sets, s. t. it is an extension of (G) fuzzy integral of point-valued function on fuzzy sets given by Wu[6]. Since the (G) fuzzy integral is a generalization of Sugeno's fuzzy integral, the paper can be also viewed as an extension of our earlier work in [7].

In the remaider of this paper, R^+ will denote the interval $[0, +\infty)$, $I(R^+)$ is the interval number set on R^+, $\tilde{R}^+$ is the fuzzy number set on R^+, X is an arbitrary fixed set, $\mathscr{A}$ is a fuzzy σ-algebra formed by the fuzzy subsets of X, $(X, \mathscr{A})$ is a fuzzy measurable space. $\mu: \mathscr{A} \to R^+$ is a finite (i. e. $g(X) < +\infty$) fuzzy measure defined by Sugeno. $\tilde{M}(X)$ denotes the set of all finite fuzzy measures on $(X, \mathscr{A})$. $\overline{F}(X)$ is the set of measurable interval-valued function, $\tilde{F}(X)$ is the set of all measurable fuzzy-valued functions. $\int_A \cdot\, d\mu$ is the (G) fuzzy integral on fuzzy sets. Other concepts and notations not defined can be found in [6, 7, 8].

2. Definitions and propositions

Definition 2.1[8]. Let F be a set-valued function, $\tilde{A}\in\tilde{\mathscr{A}}$, $\mu\in\tilde{M}(X)$. Then the (G) fuzzy integral of $\tilde{F}$ on $\tilde{A}$ with respect to μ is defined as

$$\int_{\tilde{A}} F d\mu =: \{\int_{\tilde{A}} f d\mu;\ f \in S(F)\}$$

where S(F) is the set of all measurable selections of F, " $\int_{\tilde{A}}$ " is the (G) fuzzy integral on fuzzy sets.

F is said to be integrable on $\tilde{A}$, if $\int_{\tilde{A}} F d\mu \neq \varphi$.

Proposition 2.1. Let $\tilde{f}$ be a measurable interval-valued function. Then $\tilde{f}$ is integrable on every $\tilde{A}\in\tilde{\mathscr{A}}$, $\mu(\tilde{A})<+\infty$, and

$$\int_{\tilde{A}} \tilde{f} d\mu = [\int_{\tilde{A}} f^{-} d\mu, \int_{\tilde{A}} f^{+} d\mu]$$

Definition 2.2. Let $\tilde{f}\in\tilde{F}(X)$, $\tilde{A}\in\tilde{\mathscr{A}}$. Then the (G) fuzzy integral of $\tilde{f}$ on $\tilde{A}$ is defined as,

$$(\int_{\tilde{A}} \tilde{f} d\mu)(r) =: Sup\{\lambda; r \in \int_{\tilde{A}} \tilde{f}_{\lambda} d\mu]$$

Theorem 2.1. Let $\tilde{f}\in\tilde{F}(X)$. Then $\int_{\tilde{A}} \tilde{f} d\mu \in \tilde{R}^{+}$, *and* $(\int_{\tilde{A}} \tilde{f} d\mu)_{\lambda} = \int_{\tilde{A}} \tilde{f}_{\lambda} d\mu$, where $\lambda\in(0, 1]$.

Theorem 2.2. (G) fuzzy integrals of fuzzy-valued functions on fuzzy sets have the following properties:

(i) $\tilde{f}_1 \leqslant \tilde{f}_2$ *implies* $\int_{\tilde{A}} \tilde{f}_1 d\mu \leqslant \int_{\tilde{A}} \tilde{f}_2 d\mu$;

(ii) $\tilde{A} \subset \tilde{B}$ *implies* $\int_{\tilde{A}} \tilde{f} d\mu \leqslant \int_{\tilde{B}} \tilde{f} d\mu$;

(iii) $\mu(\tilde{A}) = 0$ *implies* $\int_{\tilde{A}} \tilde{f} d\mu = \tilde{0}$, *where* $\tilde{0}(r) = \begin{cases} 1, & r = 0 \\ 0, & r \neq 0 \end{cases}$;

(iv) $\int_{\tilde{A}} \tilde{c} d\mu = S(\tilde{c}, \mu(\tilde{A}))$, $(\tilde{c} \in \tilde{R}^{+})$;

(v) $\int_{\tilde{A}} (\tilde{c} \vee \tilde{f}) d\mu = \int_{\tilde{A}} \tilde{c} d\mu \vee \int_{\tilde{A}} \tilde{f} d\mu$ $(\tilde{c} \in \tilde{R}^{+})$;

(vi) $\mu_1 \leqslant \mu_2$ *implies* $\int_{\tilde{A}} \tilde{f} d\mu_1 \leqslant \int_{\tilde{A}} \tilde{f} d\mu_2$.

Theorem 2. 3. For every $\tilde{f}_1 = \tilde{f}_2$ μ-a. e. implies $\int_{\tilde{A}} \tilde{f}_1 d\mu = \int_{\tilde{A}} \tilde{f}_2 d\mu$ iff μ is null-additive.

3. **Convergence theorems.**

Theorem 3. 1 (Generalized monotone convergence theorem).

In this section, the concept of convergence is from [7,8].

Let $\{\tilde{f}_n\} \subset \tilde{F}(X)$, $\{\mu_n (n \geqslant 1), \mu\} \subset \tilde{M}(X)$. Then

(i) $\tilde{f}_n \uparrow \tilde{f}$, $\mu_n \uparrow \mu$ *implies* $\int_{\tilde{A}} \tilde{f}_n d\mu_n \uparrow \int_{\tilde{A}} \tilde{f} d\mu$,

(ii) $\tilde{f}_n \downarrow \tilde{f}$, $\mu_n \downarrow \mu$ *implies* $\int_{\tilde{A}} \tilde{f}_n d\mu_n \downarrow \int_{\tilde{A}} \tilde{f} d\mu$.

Corollary 3. 1 (Monotone convergence theorem)

(i) $\tilde{f}_n \uparrow \tilde{f}$ implies $\int_{\tilde{A}} \tilde{f}_n d\mu \uparrow \int_{\tilde{A}} \tilde{f} d\mu$;

(ii) $\tilde{f}_n \downarrow \tilde{f}$ implies $\int_{\tilde{A}} \tilde{f}_n d\mu \downarrow \int_{\tilde{A}} \tilde{f} d\mu$;

(iii) $\mu_n \uparrow \mu$ implies $\int_{\tilde{A}} \tilde{f} d\mu_n \uparrow \int_{\tilde{A}} \tilde{f} d\mu$;

(iv) $\mu_n \downarrow \mu$ implies $\int_{\tilde{A}} \tilde{f} d\mu_n \downarrow \int_{\tilde{A}} \tilde{f} d\mu$;

Theorem 3. 2 (Generalized Fatou's lemmas). Let $\{\tilde{f}_n\} \subset \tilde{F}(X)$, $\{\mu_n\} \subset \tilde{M}(X)$. Further we assume $\{\bigvee_{k=n}^{\infty} \mu_k\}$, $\{\bigwedge_{k=n}^{\infty} \mu_k\}$ and $\{\liminf_{n\to\infty} \mu_n, \limsup_{n\to\infty} \mu_n\}$ are all included in $\tilde{M}(X)$. Then

(i) $\int_{\tilde{A}} (\liminf_{n\to\infty} \tilde{f}_n) d(\liminf_{n\to\infty} \mu_n) \leqslant \liminf_{n\to\infty} \int_{\tilde{A}} \tilde{f}_n d\mu_n$;

(ii) $(\limsup_{n\to\infty} \int_{\tilde{A}} \tilde{f}_n d\mu_n) \leqslant \int_{\tilde{A}} (\limsup_{n\to\infty} \tilde{f}_n) d(\limsup_{n\to\infty} \mu_n)$.

Corollary 3. 2 (Fatou's lemmas).

(i) If $\{\tilde{f}_n\} \subset \tilde{F}(X)$, then

a) $\int_{\tilde{A}} (\liminf_{n\to\infty} \tilde{f}_n) d\mu \leqslant \liminf_{n\to\infty} \int_{\tilde{A}} \tilde{f}_n d\mu$,

b) $\limsup_{n\to\infty} \int_{\tilde{A}} \tilde{f}_n d\mu \leqslant \int_{\tilde{A}} (\limsup_{n\to\infty} \tilde{f}_n) d\mu$.

(ii) Let $\{\mu_n\} \subset \tilde{M}(X)$. If $\{\bigvee_{k=n}^{\infty} \mu_k\}$, $\{\bigwedge_{k=n}^{\infty} \mu_k\}$ and $\{\liminf_{n\to\infty} \mu_n, \limsup_{n\to\infty} \mu_n\}$ are all included in $\tilde{M}(X)$, then

a) $\int_{\tilde{A}} \tilde{f} d(\liminf_{n\to\infty} \mu_n) \leqslant \liminf_{n\to\infty} \int_{\tilde{A}} \tilde{f} d\mu$

b) $\limsup_{n\to\infty}\int_{\tilde{A}}\mathcal{J}d\mu_n \leqslant \int_{\tilde{A}}\mathcal{J}d(\limsup_{n\to\infty}\mu_n)$.

Concluding remark.

Up to here, we have established the theory of (G) fuzzy integrals of fuzzy-valued functions on fuzzy sets, and these results generalize those of [6,7, 8]. In a subsequent paper, we will give the theory of (G) fuzzy integrals of fuzzy-valued function for fuzzy-valued fuzzy measures on fuzzy sets.

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