QUASI-CUT SETS AND OPERATIONS OF FUZZY SETS

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1. Introduction

Let X be a set and A be a fuzzy subset of X, then λ-cut set and λ-strong cut set of fuzzy set A are defined as:

$$A_{\lambda}=\{x\,|\,x\in X \text{ and } A(x)\geq\lambda\} \qquad A_{\underset{\cdot}{\lambda}}=\{x\,|\,x\in X \text{ and } A(x)>\lambda\}$$

From the point of neighborhood, we have: $x\in A_{\lambda}\longleftrightarrow A(x)\geq\lambda\longleftrightarrow x_{\lambda}\in A$. Prof.Luo Cheng-zhong has ever introduced a new concept of strong neighborhood. According to his definition[6], we have: $x\in A_{\underset{\cdot}{\lambda}}\longleftrightarrow A(x)>\lambda \longleftrightarrow x_{\lambda}\underset{\cdot}{\in} A$. In this way, we can describe λ-cut set and λ-strong cut set as:

$$A_{\lambda}=\{x\,|\,x_{\lambda}\in A\} \qquad A_{\underset{\cdot}{\lambda}}=\{x\,|\,x_{\lambda}\underset{\cdot}{\in} A\}$$

It is well known that quasi-neigbborhood play an important role in fuzzy topology. According to [1] we have:

$x_{\lambda}qA$(it is denoted as $x_{\lambda}\in_{q}A)\longleftrightarrow\lambda+A(x)>1$

From the point of quasi-neighborhood, we can define a new kind of cut set as:

$$A_{[\underset{\cdot}{\lambda}]}=\{x\,|\,x_{\lambda}\in_{q}A\}$$

$A_{[\underset{\cdot}{\lambda}]}$ is called as λ-strong quasi-cut set of fuzzy set A.

In this paper, we shall discuss some properties of quasi-cut set and give decomposition theorem and representation theorem based on quasi-cut sets. As applications, we shall give a new approach to define operations of fuzzy sets based on quasi-cut sets and Wang's falling shadows theory and obtain the same results as [2].

2. Theory of falling shadow and marginal-uniform joint distribution on $[0,\ 1]^2$

Given a universe of discourse X, $\mathbb{P}(X)$ denotes as power set of X. For any $x\in X$, let

$$\dot{x}=\{A\,|\,x\in A \text{ and } A\in\mathbb{P}(X)\},\quad \dot{A}=\{\dot{x}\,|\,x\in A\}$$

An ordered pair $(\mathbb{P}(X),\mathbb{B})$ is said to be hyper-measurable structure on X if $\mathbb{B}$ is σ-field in $\mathbb{P}(X)$ and $\dot{X}\subseteq\mathbb{B}$.

Given a probability space $(\Omega,\mathbb{A},P)$ and an hyper-measurable structure $(\mathbb{P}(X),\mathbb{B})$ on X, a random set on X is defined to be a mapping $\xi:\Omega\longrightarrow\mathbb{P}(X)$ that is $\mathbb{A}$-$\mathbb{B}$ measurable, that is

$$\forall C\in\mathbb{B},\ \xi^{-1}(C)=\{\omega|\omega\in\Omega \text{ and } \xi(\omega)\in C\}\in\mathbb{A}$$

Suppose that ξ is a random set on X. Then the covering function of ξ, denoted by $\hat{\xi}$, is defined to be the probability of ω for which $x\in\xi(\omega)$, that is, $\hat{\xi}:X\longrightarrow[0,\ 1]$ and $\hat{\xi}(x)=P(\omega|x\in\xi(\omega))$, for each $x\in X$. $\hat{\xi}$ represents a fuzzy set A in X and we write $\hat{\xi}=A$. We shall call the random set ξ a cloud on A, and A the falling shadows of the random set ξ.

Once the probability space has been determined, then to each fuzzy set A in X, there corresponds a family of random sets whose falling shadows are all equal to A. Thus it is an important issue on how to choose a representative cloud for A. Indeed, the simplest one is the mapping $\xi:[0,\ 1]\longrightarrow\mathbb{P}(X)$ $\lambda\longrightarrow A_\lambda$. where A_λ is the λ-cut of A, that is $A_\lambda=\{x\in X|A(x)\geq\lambda\}$.

Let $\mathbb{B}$ is a Borel field on [0, 1] and m is a Lebesgue measure. Let joint probability $([0,\ 1]^2,\mathbb{B}^2,P)=([0,\ 1],\mathbb{B},m)\times([0,\ 1],\mathbb{B},m)$, where $P(A\times[0,\ 1])=m(A)$, $P([0,\ 1]\times A)=m(A)$, for each $A\in\mathbb{B}$, then there are infinitely many possibilities of joint distribution of P on the unit square $[0,\ 1]^2$. We shall refer P to as a marginal-uniform joint distribution on $[0,\ 1]^2$ and consider the following three basic cases.

I. *Perfect positive correlation*

If the whole probability P of (λ,μ) on $[0,\ 1]^2$ is concentrated and uniformly distributed on the main diagonal $I=\{(\lambda,\lambda)|\lambda\in[0,\ 1]\}$ of the unit square $[0,\ 1]^2$, then we say the variables λ and μ are in perfect positive correlation.

II. *Perfect negative correlation*

If the whole probability P of (λ,μ) on $[0,\ 1]^2$ is concentrated and uniformly distributed on the anti-diagonal $\hat{I}=\{(\lambda,1-\lambda)|\lambda\in[0,\ 1]\}$ of the unit square $[0,\ 1]^2$, then we say that the variables λ and μ are in perfect negative correlation.

III. *Independent*

If the whole probability P of (λ,μ) on $[0,\ 1]^2$ is uniformly distributed on the unit square $[0, 1]^2$, then we say that the variables λ and μ are independent.

3. Definition and properties of quasi-cut set

Definition 3.1 Let A be a fuzzy subset of set X, then

$$A_{[\lambda]}=\{x \mid x\in X, \lambda+A(x)\geq 1\},\quad A_{[\underset{\cdot}{\lambda}]}=\{x \mid x\in X, \lambda+A(x)>1\}$$

are called as λ-quasi-cut set and λ-strong quasi-cut set of fuzzy set A respectively.

Definition 3.2 Let X be a set and $H:[0\ 1]\longrightarrow \mathcal{P}(X)$, $\lambda\longrightarrow H(\lambda)$ satisfy: $\lambda_1<\lambda_2 \Longrightarrow H(\lambda_1)\subseteq H(\lambda_2)$, then H is called as a order nested set over X.

Clearly, $H(\lambda)=A_{[\lambda]}$ or $A_{[\underset{\cdot}{\lambda}]}$ is a order nested set over X respectively.

Theorem 3.1 (1) $(A\cup B)_{[\lambda]}=A_{[\lambda]}\cup B_{[\lambda]}$, $(A\cup B)_{[\underset{\cdot}{\lambda}]}=A_{[\underset{\cdot}{\lambda}]}\cup B_{[\underset{\cdot}{\lambda}]}$, $(A\cap B)_{[\lambda]}=A_{[\lambda]}\cap B_{[\lambda]}$, $(A\cap B)_{[\underset{\cdot}{\lambda}]}=A_{[\underset{\cdot}{\lambda}]}\cap B_{[\underset{\cdot}{\lambda}]}$

(2) $\lambda_1<\lambda_2 \Longrightarrow A_{[\lambda_1]}\subseteq A_{[\lambda_2]}$, $A_{[\lambda_1]}\subseteq A_{[\lambda_2]}$, $A_{[\underset{\cdot}{\lambda}_1]}\subseteq A_{[\lambda_2]}$

(3) $(\bigcup_{t\in T}A^t)_{[\lambda]}\supseteq \bigcup_{t\in T}A^t_{[\lambda]}$, $(\bigcup_{t\in T}A^t)_{[\underset{\cdot}{\lambda}]}=\bigcup_{t\in T}A^t_{[\underset{\cdot}{\lambda}]}$, $(\bigcap_{t\in T}A^t)_{[\lambda]}=\bigcap_{t\in T}A^t_{[\lambda]}$, $(\bigcap_{t\in T}A^t)_{[\underset{\cdot}{\lambda}]}\subseteq\bigcap_{t\in T}A^t_{[\underset{\cdot}{\lambda}]}$

(4) $(A_{[\lambda]})^c=(A^c)_{[1-\lambda]}$, $(A_{[\underset{\cdot}{\lambda}]})^c=(A^c)_{[1-\lambda]}$

(5) $A_{[\bigvee_{t\in T}\alpha_t]}\supseteq \bigcup_{t\in T}A_{[\alpha_t]}$, $A_{[\bigwedge_{t\in T}\alpha_t]}=\bigcap_{t\in T}A_{[\alpha_t]}$, $A_{[\bigvee_{t\in T}\alpha_t]}=\bigcup_{t\in T}A_{[\alpha_t]}$, $A_{[\bigwedge_{t\in T}\alpha_t]}\subseteq\bigcap_{t\in T}A_{[\alpha_t]}$

4. Decomposition theorem and representation theorem based on quasi-cut sets.

Let C be a subset of set X and $\lambda\in[0,1]$, we define λC as a fuzzy subset of X and

$$(\lambda C)=\begin{cases}\lambda, & x\in C\\ 0, & x\notin C\end{cases}$$

then we have

Theorem 4.1 (1) $A=\bigcup_{\lambda\in[0,1]}\lambda^c A_{[\lambda]}$, (2) $A=\bigcup_{\lambda\in[0,1]}\lambda^c A_{[\underset{\cdot}{\lambda}]}$, (3) Let $H:[0, 1]\longrightarrow (X)$ satisfy: $A_{[\underset{\cdot}{\lambda}]}\subseteq H(\lambda)\subseteq A_{[\lambda]}$, then (i) $\lambda_1<\lambda_2\Longrightarrow H(\lambda_1)\subseteq H(\lambda_2)$; (ii) $A=\bigcup_{\lambda\in[0,1]}\lambda^c H(\lambda)$ (iii) $A_{[\lambda]}=\bigcap_{\alpha>\lambda}H(\alpha)$, $A_{[\underset{\cdot}{\lambda}]}=\bigcup_{\alpha<\lambda}H(\alpha)$

Let $\mathbb{U}(X)$ be set of order nested sets over X, we define operations $\bigcup, \bigcap$, on $\mathbb{U}(X)$ as follwing:

$$\bigcup_{r\in\Gamma} H_r: (\bigcup_{r\in\Gamma} H_r)(\lambda)=\bigcup_{r\in\Gamma} H_r(\lambda), \quad \bigcap_{r\in\Gamma} H_r: (\bigcap_{r\in\Gamma} H_r)(\lambda)=\bigcap_{r\in\Gamma} H_r(\lambda), \quad H^c: H^c(\lambda)=(H(1-\lambda))^c$$

then we have:

Theorem 4.2 Let $\mathbb{F}(X)$ be a set of fuzzy subset of X on [0, 1]. Let

$$T:\mathbb{U}(X)\longrightarrow \mathbb{F}(X) \quad T(H)=\bigcup_{\lambda\in[0,\ 1]}\lambda^c H(\lambda), \text{ then}$$

(1) $T(H)_{\underset{\cdot}{[\lambda]}}\subseteq H(\lambda)\subseteq T(H)_{[\lambda]}$, (2) $T(H)_{[\lambda]}=\bigcap_{\alpha>\lambda}H(\alpha)$, $T(H)_{\underset{\cdot}{[\lambda]}}=\bigcup_{\alpha<\lambda}H(\alpha)$

(3) T is a homomorphism from $(\mathbb{U}(X),\bigcup,\bigcap,c)$ to $(\mathbb{F}(X),\bigcup,\bigcap,c)$, i.e.

(i) For any $A\in\mathbb{F}(X)$ there is a $H\in\mathbb{U}(X)$ such that $T(H)=A$;

(ii) $T(\bigcup_{r\in\Gamma} H_r)=\bigcup_{r\in\Gamma} T(H_r)$, $T(\bigcap_{r\in\Gamma} H_r)=\bigcap_{r\in\Gamma} T(H_r)$, $T(H^c)=T(H)^c$

5. Operations of fuzzy sets based on falling shadows theory and quasi-cut sets

Let A and B be fuzzy sets in the universe X. Let $\mathbb{B}$ a Borel field on [0, 1] and m be a Lebesgue measure, the joint probability space is $([0,\ 1]^2, \mathbb{B}^2, P)$ and both of the projection of P on the [0, 1] are the Lebesgue measure m.

Theorem 5.1 Let $\xi:[0,\ 1]\longrightarrow \mathbb{P}(X)$ $\lambda\longrightarrow A_{[\lambda]}$ be a random set over X, then

$$A(x)=m(\lambda\,|\,\lambda\in[0,1],x\in A_{[\lambda]}) \qquad (1)$$

In [2], by the use of falling shadow theory and cut sets, a theoretical approach to define operations of fuzzy sets is built. In this part, we shall rebuild this approach based on falling shadow theory and quasi-cut sets.

Let $\xi:[0\ 1]^2$ $\qquad \eta:[0\ 1]^2$

$$(\lambda,\mu)\longrightarrow A_{[\lambda]} \qquad (\lambda,\mu)\longrightarrow B_{[\lambda]} \qquad (2)$$

be random sets over X, we define:

$$\xi\bigcup\eta: (\xi\bigcup\eta)(\lambda,\mu)=\xi(\lambda,\mu)\bigcup\eta(\lambda,\mu)=A_{[\lambda]}\bigcup B_{[\mu]}$$

$$\xi\bigcap\eta: (\xi\bigcap\eta)(\lambda,\mu)=\xi(\lambda,\mu)\bigcap\eta(\lambda,\mu)=A_{[\lambda]}\bigcap B_{[\mu]}$$

then $\xi\bigcup\eta$ and $\xi\bigcap\eta$ are also random sets over X, then we have:

Definition 5.1 $A\bigcup B$ and $A\bigcap B$ are defined as falling shadows of $\xi\bigcup\eta$ and $\xi\bigcap\eta$ respectively, i.e.

$$(A\bigcup B)(x)=P((\lambda,\mu)\,|\,x\in A_{[\lambda]}\bigcup B_{[\mu]}), \quad (A\bigcap B)(x)=P((\lambda,\mu)\,|\,x\in A_{[\lambda]}\bigcap B_{[\mu]})$$

Let

$E_1=[0,\ A^c(x)]\times[0,\ B^c(x)],\ E_2=[0,\ A^c(x)\times[B^c(x),\ 1]$

$E_3=[A^c(x),\ 1]\times[B^c(x),\ 1],\ E_4=[A^c(x),\ 1]\times[0,\ B^c(x)]$

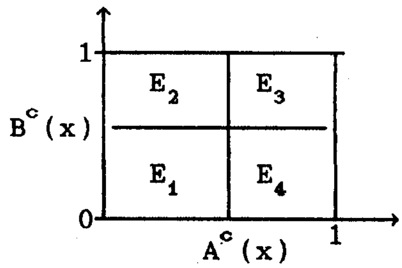

Fig.1

then we have :

Theorem 5.2 $(A\cup B)(x)=P(E_2\cup E_3\cup E_4),\ (A\cap B)(x)=P(E_3)$ (3)

Theorem 5.3 (i). If the fuzzy sets A and B are in perfect positive correlation, then the formula (3) will become:

$(A\cup B)(x)=\max\{A(x),B(x)\},\ (A\cap B)(x)=\min\{A(x),B(x)\}$ (4)

(ii) If the fuzzy sets A and B are in perfect negative correlation, then formula (3) will become:

$(A\cup B)(x)=\min\{A(x)+B(x),1\},\ (A\cap B)(x)=\max\{A(x)+B(x)-1,0\}$ (5)

(iii) If the fuzzy sets A and B are independent, then formula (3) will become:

$(A\cup B)(x)=A(x)+B(x)-A(x)B(x),\ (A\cap B)(x)=A(x)B(x)$ (6)

Refernces

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