THE DEGREE OF CONSISTENCY BETWEEN THE POSSIBILITY AND PROBABILITY

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Abstract

In present paper,we first introduce the concepts of the consistency distribution between the possibility distribution and probability distribution ,the noninteractional consistency distribution and the orthogonal consistency distribution.Then some basic properties are discussed.Next step,the degree of the consistency,the conditional degree of the consistency,and the independence of consistency distribution are defined in turn. And some properties of interest are also our topics.

Keywords: Possibility distribution ,Probability distribution, Consistency distribution, The degree of consistency.

1.Preliminaries

Let X be an at most countable set, mapping

$$p : X \rightarrow [0,1]$$

is a probability distribution on X.

$$\pi : X \rightarrow [0,1]$$

is a possibility distribution.It is called normal possibility distribution when $\bigvee_{x \in X} \pi(x) = 1$

Let PS(X) denote the family of normalized possibility distribution over X, PR(X) denote the family of probability distributions over X. We know the simultaneous existence of both kind of information about X, a probability and possibility distribution, and the question about the relation between them arises at once. For these situation Zadeh [6] established the possibility-probability consistency principle: the degree of consistency between the possibility distribution and the probability distribution is expressed by

$$C_Z(\pi, p) = \sum_{x\in X} \pi(x)p(x)$$

Definition1.1: Let $T: [0,1] \times [0,1] \to [0,1]$ be a function, T is called a t-seminorm, if and only if it fulfills the properties:

(1) $T(1,x) = T(x,1) = x$, for all x of $[0,1]$.

(2) If $x_1 \leq x_2$ and $y_1 \leq y_2$ then $T(x_1,y_1) \leq T(x_2,y_2)$.

Clearly, $T_0(x,y) = xy$, $T_1(x,y) = x \wedge y$ are t-seminorm.

Definition1.2: Let $T : [0,1] \times [0,1] \longrightarrow [0,1]$ be a function, T is called a t-semiconorm, if and only if it fulfills the properties:

(1) $\perp(x,o) = \perp(0,x) = x$, for all x of $[0,1]$

(2) If $x_1 \leq x_2$ and $y_1 \leq y_2$ then $\perp(x_1,y_1) \leq \perp(x_2,y_2)$.

It is easy to see that $\perp$ is a t-semiconorm if and only if there exist a t-seminorm T such that $\perp(x,y) = 1 - T(1-x,1-y)$.

Clearly, $\perp_0(x,y)=x+y$, $\perp_1(x,y)=x\vee y$ are t-semiconorm.

The t-norm and t-conorm concepts of Schweizer and Sklar, see [5], turn out to be special cases of the previous definitions.

2. Consistency distribution

Definition2.1: Let $\pi \in PS(X)$, $p \in PR(X)$, mapping

$C(\pi(x),p(x)) : \pi \times p \longrightarrow [0,1]$

$C(\pi(x),p(x)) \triangleq T(\pi(x),P(x))$

is called consistency distribution over X between possibility distribution π and probability distribution p. Where T is t-seminorm.

Clearly, $C_Z(\pi,p) = \sum_{x\in X} T_0(\pi(x),p(x))$.

Let $\mathcal{E}(X)$ denote the family of consistency distribution over X. and $C(\pi(x),p(x)) = C(\pi,p)$ (or briefly C).

Proposition2.1: Consistency distribution have the following properties

(1) $\lambda\in[0,1]$, $C_1, C_2 \in \mathcal{E}(X) \Longrightarrow \lambda C_1 + (1-\lambda)\cdot C_2 \in \mathcal{E}(X)$.

(2) $C_1\in \mathcal{E}(X)$, $C_2\in \mathcal{E}(X) \Longrightarrow$ $\mathrm{Min}(C_1,C_2)\in \mathcal{E}(X)$, $\mathrm{Max}(C_1,C_2)\in \mathcal{E}(X)$.

(3) $C'\in \mathcal{E}(X)$, $g(s)$: $[0,1]\longrightarrow[0,1]$ is a strictly monotone increasing continuous function with $g(0)=0$, $g(1)=1$, and $G(s)$ is inverse function of $g(s)$, then

$$C(\pi,p) = G(C'(g(\pi),g(p)))\in \mathcal{E}(X).$$

Besides, if

$$C'(C'(a,b),c) = C'(a,C'(b,c))$$

then

$$C(C(a,b),c) = C(a,C(b,c))$$

Proof: Straightforward.

Proposition2.2: Let $C(\pi_1,p)\in \mathcal{E}(X)$, $C(\pi_2,p)\in \mathcal{E}(X)$, if for any $a\in X$, $\pi_1(a) \geqq \pi_2(a)$, then $C(\pi_1,p) \geqq C(\pi_2,p)$.

Proposition2.3: Let $C(\pi,p_1)\in \mathcal{E}(X)$. $C(\pi,p_2)\in \mathcal{E}(X)$, if for any $a\in X$, $p_1(a) \geqq p_2(a)$, then $C(\pi,p_1) \geqq C(\pi,p_2)$.

The proof of proposition2.2 and 2.3 are obvious.

Let $X = \{x_i \mid i\in N\}$, for any $\pi\in PS(X)$, $p\in PR(X)$, it is clear that $\{\pi(x)\}$ and $\{p(x)\}$ are at most contable sets. If $C\in \mathcal{E}(X)$, then $\{C(\pi,p)\}$ is also an at most contable set. Let Σ_1 be a σ-algebra of subsets of $\Omega_1=$

$\{\pi(x_i)\mid i\in N\}$ and Σ_2 a σ-algebra of subsets of $\Omega_2=\{p(x_i)\mid i\in N\}$, then $\Sigma=\Sigma_1\times\Sigma_2$ is a σ-algebra of subsets of $\Omega=\Omega_1\times\Omega_2$.

Proposition2.4: Let $\pi\in$ PS(X), $p\in$ PR(X), then

$$\mathcal{B}=\{x\mid(\pi,p)\in\Sigma\}$$

is a σ-algebra on X.

Proof: First, since $\Omega_1\in\Sigma_1$, $\Omega_2\in\Sigma_2$, hence

$$X=\{x\mid \pi(x)\in\Omega_1,\quad \forall x\in X\}$$
$$=\{x\mid p(x)\in\Omega_2,\quad \forall x\in X\}$$
$$=\{x\mid \pi(x)\in\Omega_1,\ p(x)\in\Omega_2,\ \forall x\in X\}$$

Besides

$$\{(\pi,p)\mid \pi(x)\in\Omega_1,\ p(x)\in\Omega_2,\ \forall x\in X\}\in\Sigma.$$

that is

$$\{(\pi,p)\mid (\pi,p)\in\Omega\}\in\Sigma$$

therefore

$$X=\{x\mid (\pi,p)\in\Omega\}\in\mathcal{B},\quad (\text{because } \Omega\in\Sigma)$$

Next, let $A\in\mathcal{B}$, then

$$\{(\pi,p)\mid x\in A\}\in\Sigma$$

because

$$\{(\pi,p)\mid (\pi,p)\in\Omega\}\in\Sigma$$

hence

$$\{(\pi,p)\mid (\pi,p)\in\Omega\}-\{(\pi,p)\mid x\in A\}\in\Sigma.$$

then, we have

$$\bar{A}=X-A=\{x\mid\{(\pi,p)\mid(\pi,p)\in\Omega\}-\{(\pi,p)\mid x\in A\}\}\in\mathcal{B}.$$

Finally, let $B_i\in\mathcal{B}$, $i\in N$, owing to Σ is a σ-algebra, hence we have

$$\bigcup_{i=1}^{\infty}\{(\pi,p)\mid x\in B_i\}\in\Sigma.$$

and

$$\left\{ x \;\middle|\; \bigcup_{i=1}^{\infty}\{(\pi,p) \mid x\in B_i\}\in\Sigma \right\}\in\mathcal{B}$$

that is

$$\bigcup_{i=1}^{\infty} B_i \in \mathcal{B} .$$

Therefore $\mathcal{B}$ is a σ-algebra on X. The proof is finished.

$(X,\mathcal{B})$ is called a measurable space.

Definition2.2: $C\in \mathcal{C}(X)$ is said to be Σ-measurable, if and only if

$$\{(\pi,p) \mid C(\pi,p)\in[0,\alpha]\}\in\Sigma$$

for each $\alpha\in[0,1]$.

Proposition2.5: Let C_1 and C_2 are Σ-measurable consistency distribution, then $C_1\vee C_2$ and $C_1\wedge C_2$ are also Σ-measurable consistency distributions.

Proof: Using proposition2.1 (2), we may get $C_1\vee C_2$ and $C_1\wedge C_2$ are consistency distribution. The next only to show they are Σ-measurable.

Since X is an at most countable set, hence $\{C_1\vee C_2\}$ is also an at most countable set, therefore, for each $\alpha\in[0,1]$, we have

$$\{(\pi,p) \mid C_1\vee C_2\in[0,\alpha]\} = \bigcup_k \{(\pi_{ik},p_{ik})\}$$

where $\pi_{ik}=\pi(x_{ik})$, $p_{ik}=p(x_{ik})$.

We denote

$$C_1(\pi_{ik}, p_{ik})\vee C_2(\pi_{ik}, p_{ik}) = m_k ,$$

$$M_k = \{(\pi,p) \mid C_1\vee C_2\in[0,m_k]\}$$

obviously,

$$0\le m_k\le\alpha , \quad M_k = \bigcup_j \{(\pi_{ikj}, p_{ikj})\}$$

Besides, owing to

$$C_1(\pi_{ikj}, p_{ikj})\vee C_2(\pi_{ikj}, p_{ikj}) = \begin{cases} C_1(\pi_{ikj}, p_{ikj}), & C_1\ge C_2 \\ C_2(\pi_{ikj}, p_{ikj}), & C_1\not\ge C_2 \end{cases}$$

and C_1, C_2 are Σ-measurable, hence

$$\{(\pi_{ikj}, p_{ikj})\} \subset \{(\pi,p) \mid C_t(\pi_{ikj}, p_{ikj}) \in [0, m_k], \quad t=1,2\} \in \Sigma$$

Therefore

$$M_k = \bigcup_j \{(\pi_{ikj}, p_{ikj})\} \in \Sigma$$

and

$$\{(\pi,p) \mid C_1 \vee C_2 \in [0,\alpha]\} = \bigcup_k M_k \in \Sigma$$

That is $C_1 \vee C_2$ is Σ-measurable. For the case of $C_1 \wedge C_2$ one may give a similar proof.

Definition2.3: Let $\pi \in PS(X)$, $p \in PR(X)$, consistency distribution C is said to be noninteraction, if and only if

$$C(\pi(x), p(x)) = T_1(\pi(x), p(x)) \qquad \forall x \in X$$

In this time, we also called that possibility distribution π and probability distribution p are noninteraction.

Definition2.4: Let $\pi \in PS(X)$, $p \in PR(X)$, consistency distribution C is said to be orthogonal consistency distribution, if and only if

$$C(\pi(x), p(x)) = \bot(\pi(x), P(x)) \qquad \forall x \in X$$

In this time, we also called that possibility distribution π and probability distribution p are orthogonal.

It is easy to see that if $\pi(x) = 0$, then for any $p(x) \in PR(X)$, $C(\pi,p) = \bot(\pi,p) = p$; if $p(x) = 0$, then for any $\pi(x) \in PS(X)$, $C(\pi,p) = \bot(\pi,p) = \pi$. In this both case, the possibility distribution π and the probability distribution p are called strict orthogonal.

Definition2.5: $\pi(x) \in PS(X)$ is called $\mathscr{E}$-measurable, if and only if for any $\alpha \in [0,1]$

$$\{x \mid \pi(x) \in [0,\alpha]\} \in \mathscr{E}$$

Definition2.6: $p(x) \in PR(X)$ is called $\mathscr{B}$-measurable, if and only if for any $\alpha \in [0,1]$

$$\{x \mid p(x) \in [0,\alpha]\} \in \mathscr{B}$$

Proposition2.6: If consistency distribution $C(\pi,p)$ is noninteraction and Σ-measurable, then $\pi(x)$ ($p(x)$) is $\mathscr{B}$-measurable.

Proof: Since $\{\pi(x)\}$ is an at most countable set, hence for each $\alpha \in [0,1]$, we have

$$\{x \mid \pi(x) \in [0,\alpha]\} = \bigcup_k \{x_{ik}\}$$

We denote $\pi(x_{ik}) \wedge p(x_{ik}) = l_k$, that is $C(\pi(x_{ik}), p(x_{ik})) = l_k$, obviously, $0 \leq l_k \leq 1$. Denote $M_k = \{(\pi,p) \mid C(\pi,p) \in [0,l_k]\}$, owing to $C(\pi,p)$ is Σ-measurable, hence $M_k \in \Sigma$, using proposition2.4, we have

$$\{x_{ik}\} \subset \{x \mid (\pi,p) \in M_k\} \in \mathscr{B}$$

therefore

$$\bigcup_k \{x_{ik}\} \subset \bigcup_k \{x \mid (\pi,p) \in M_k\} \in \mathscr{B}$$

that is

$$\{x \mid \pi(x) \in [0,\alpha]\} \in \mathscr{B}$$

Hence $\pi(x)$ is $\mathscr{B}$-measurable, similar, $p(x)$ is also $\mathscr{B}$-measurable.

Proposition2.7: Let $\pi(x) \in PS(X)$, $p(x) \in PR(X)$, then

$$C(\pi,p) = 1 - \perp(1-\pi, 1-p) \in \mathscr{C}(X)$$

Proof: For this we have only to show that $C(\pi,p) = T(\pi,p)$.

$$\begin{aligned} C(1,p(x)) &= 1 - \perp(0, 1-p(x)) \\ &= 1-(1-p(x)) \\ &= p(x) . \end{aligned}$$

$$\begin{aligned} C(\pi(x),1) &= 1 - \perp(1-\pi(x), 0) \\ &= 1-(1-\pi(x)) \\ &= \pi(x) . \end{aligned}$$

If $\pi(x_1) \leq \pi(x_2)$, $p(y_1) \leq p(y_2)$

then

$$1-\pi(x_1) \geq 1-\pi(x_2)\ , \quad 1-p(y_1) \geq 1-p(y_2)$$

hence

$$\perp(1-\pi(x_2)\ ,\ 1-p(y_2)) \leq \perp(1-\pi(x_1)\ ,1-p(y_1))$$

so

$$1-\perp(1-\pi(x_2)\ ,1-p(y_2)) \geq 1-\perp(1-\pi(x_1),1-p(y_1))$$

that is

$$C(\pi(x_1),p(y_1)) \leq C(\pi(x_2),P(y_2))$$

Therefore $C(\pi,p)\in \mathcal{E}(X)$.The proof is finished.

3. The degree of consistency

Definition3.1: Let $(X,\mathcal{B})$ is a measurable space,$C\in\mathcal{E}(X)$ is Σ-measurable, we define

$$P(C) = \int_A C(\pi(x),p(x))dP$$

and name the quantity the degree of consistency between the possibility distribution π and probability distribution p.

Where $P(A) = \sum_{x\in A} p(x) \qquad \forall\ A\in\mathcal{B}$

It is analogous to the Zadeh's probability of fuzzy events,we have

1. If $C\in\mathcal{E}(X)$ is Σ-measurable ,then $0 \leq P(C) \leq 1$.
2. If $C(\pi,p) = 1$, then $P(C) = 1$.
3. If $C_1(\pi,p) \leq C_2(\pi,p)$,then $P(C_1) \leq P(C_2)$.

Proposition3.1: Let $C_1\in\mathcal{E}(X)$,$C_2\in\mathcal{E}(X)$ are Σ-measurable,then

$$P(C_1\vee C_2) = P(C_1) + P(C_2) - P(C_1\wedge C_2)$$

Proof: Since

$$\begin{aligned}P(C_1\vee C_2)+P(C_1\wedge C_2) &= \int_X(C_1\vee C_2)dP + \int_X (C_1\wedge C_2)dP\\ &= \int_X\left[(C_1\vee C_2) + (C_1\wedge C_2)\right]dP\\ &= \int_X (C_1+C_2)dP\\ &= \int_X C_1dP+\int_X C_2dP\end{aligned}$$

$$= P(C_1) + P(C_2)$$

Hence

$$P(C_1 \vee C_2) = P(C_1) + P(C_2) - P(C_1 \wedge C_2).$$

Corollary3.1: If $C_1 \wedge C_2 = 0$, then

$$P(C_1 \vee C_2) = P(C_1) + P(C_2).$$

Proposition3.2: If $C \in \mathcal{E}(X)$ is Σ-measurable, then

$$P(1-C) = 1 - P(C)$$

Proof: Obvious.

Definition3.2: Let $\pi(x) \in PS(X)$ is $\mathcal{B}$-measurable, for arbitrary $p(x) \in PR(X)$, the marginal degree of consistency about π is defined by

$$P(\pi) = \int_X \pi(x) dP$$

Proposition3.3: If $C(\pi, p) \in \mathcal{E}(X)$ is noninteraction and Σ-measurable, then

$$P(C) \leq P(\pi)$$

Proof: This is evident.

Proposition3.4: If $\pi \in PS(X)$ and $p \in PR(X)$ are strict orthogonal, and π, p $\mathcal{B}$-measurable, then

$$P(C) = \begin{cases} P(\pi) & \pi(x) \neq 0 \\ 1 & \pi(x) = 0 \end{cases}$$

Proof: Obvious.

4. Conditional degree of consistency and independent

Definition4.1: Let $C_1 \in \mathcal{E}(X)$ and $C_2 \in \mathcal{E}(X)$ are Σ-measurable, if $P(C_2) > 0$

$$P(C_1 | C_2) \triangleq \frac{P(C_1 \cdot C_2)}{P(C_2)}$$

is called coditional degree of consistency about C_2.

Definition4.2: Let $C_1 \in \mathcal{E}(X)$ and $C_2 \in \mathcal{E}(X)$ are Σ-measurable, C_1 and C_2 are said to be independent each other, if and only if

$$P(C_1 \cdot C_2) = P(C_1) \cdot P(C_2)$$

It is analogous to the conditional probability, we have

1. If $C_1 = 1$, then $P(C_1 | C_2) = 1$
2. $0 \leq P(C_1 | C_2) \leq 1$
3. $P(C_1 + C_2 | C_3) = P(C_1 | C_3) + P(C_2 | C_3)$

Proposition4.1: If $P(C_2) > 0$, then $C_1 \in \mathcal{E}(X)$ and $C_2 \in \mathcal{E}(X)$ are independent if and only if $P(C_1 | C_2) = P(C_1)$.

Proof: This is evident.

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