Fuzzy Logic as Applied Linear Logic

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Abstract

In common sense reasoning, logical connectives such as "and" have two different meanings. For example, if A stands for "I have a dollar", B for "I can buy a can of coke", C for "I can buy a cookie", and let coke and cookie cost $1 each. Then, $A \to B$ and $A \to C$. In 2-valued logic, we can conclude that $A \to (B\&C)$, but, since our resources are limited, with $1, we cannot buy both a coke *and* a cookie. We need two dollars to buy *both*, which can be expressed as $(A\&A) \to (B\&C)$. (Here, clearly, $A\&A \neq A$.)

To formalize these two meaning, a new logic with several connectives for "and", "or", etc., was proposed in 1987 called *linear logic*. In this paper, we show that from the algebraic viewpoint, fuzzy logic can be viewed as an important particular case of linear logic. For this particular case, we find the explicit expressions for new logical operations proposed by linear logic.

1 A Brief Introduction

To describe commonsense reasoning, a new logic was proposed in 1987 called *linear logic*. Although this logic intends to describe the same reasoning as *fuzzy logic*, these two logics were so far largely unrelated. In this paper, we show that in some reasonable sense, fuzzy logic can be viewed as a particular case of linear logic.

2 What is Linear Logic

Some researchers believe that only formalisms with $A\&A = A$ can be called *logics*, because from the common sense viewpoint, $A\&A$ means the same as A. On the other hand, if we have two independent statements A and A' with equal degrees of belief $d(A) = d(A') = a$, then the degree of belief that *both* statements are true is smaller than the degree of belief a that *one* of them is true: $f_{\&}(a,a) < a$; hence, such formalisms adequately describe human reasoning and therefore, deserve the name of "logic".

There exists a logic (called *linear*) in which $A\&A \neq A$ [2, 7]. The following example explains its idea: let A stands for "I have a dollar", B for "I can buy a coke", C for "I can buy a cookie", and let coke and cookie cost \$1 each. Then, $A \to B$ and $A \to C$. In 2-valued logic, we can conclude that $A \to (B\&C)$, but, since our resources are limited, with \$1, we cannot buy both a coke *and* a cookie. We need two dollars to buy both, which can be expressed as $(A\&A) \to (B\&C)$. Here, clearly, $A\&A \neq A$.

In traditional 2-valued logic, since $A\&A$ mean the same thing as A, in a proof, we can use any premise A as many times as we want. As a result, if we represent the proof graphically, the original premise A may *branch* into several possible branches that correspond to uses of A in different parts of the proof (e.g., in the above example, A is used to prove B and to prove C in the proof of $B\&C$). As a result, we get a *tree*. In resource-bounded logic, we cannot use a premise twice, so, we cannot branch, and the proofs become *linear* (hence the name of this logic).

Linear logic was originally proposed to describe such bounded resources as the number of available processors, but degree of belief can also be viewed as a resource [1]: For example, in 2-valued logic, from the statement "all birds fly" ($\forall b\, F(b)$), we can conclude that $F(b_1)\&\ldots\&F(b_n)$ for any number of birds n. If we only have some degree of belief in $\forall b\, F(b)$, then we may be able to believe that $F(b_1)$ for a "randomly" picked bird b_1, but our degree of belief that, say, 10^6 birds located in some area all fly will be much smaller, almost at the level of disbelief.

An interesting feature of linear logic is that it has *two* different connectives to describe the commonsense statement "B and C": "both", meaning that we can have both conclusions, and "and", meaning that we can have B, and we can have C, but not necessarily both. For "both", we have A"both"$A \neq A$; for "and", we have A"and"$A = A$. Crudely speaking, "and" correspond to $\min(a,b)$, while "both" looks more like $a \cdot b$.

Comment. As we have noticed in [4], the idea of this fine distinction is not completely alien to fuzzy logic: namely, this distinction may explain the necessity to use several different $\&-$operations in fuzzy control [6]: as the control situation changes, we are *not* changing the way we think (that would be impossible), we are just changing the meaning of the word *and*. In [4], we simply described this as an idea. In this paper, we show that the two formalisms (of linear and fuzzy logic) can be naturally combined. We hope that this combination will be helpful to both logics.

Similarly to "and", other logical connectives have several different representations in this logic.

In addition to these connectives, linear logic has a special connective $!A$ ("absolutely A") that, crudely speaking, corresponds to the absolute degree of belief in A; if $!A$ is true (i.e., if A is absolutely true), then we can use this statement A as many times as we want. In other words, in contrast to the fact that in general, in linear logic, $A\&A \neq A$, the conjunction $(!A)\&(!A)$ is equivalent to $!A$.

3 An Algebraic Approach to Linear Logic: Idea

Logics are often described in *algebraic* terms: Namely, if we have a theory based on a given logic, then we say that two statements S and S' from this theory *have the same degree of belief* if in this theory, we can prove that S is equivalent to S' (i.e., that S implies S' and that S' implies S). Thus, we can define a *degree of belief* as a class of all formulas that are equivalent to each other. Logical operations are usually consistent with this equivalence: namely, if S is equivalent to S', and T is equivalent to T', then $S\&T$ is equivalent to $S'\&T'$. As a result, each logical connective becomes an operation on the set X of degrees of belief: namely, if $x, y \in X$, i.e., if x and y are classes of equivalent statements, then we can take any statements $S \in x$ and $T \in y$ and define $x\&y$ as a class of statement that are equivalent to $S\&T$.

In particular, in 2-valued logic, if we take a *complete* theory (i.e., a theory in which every statement is either provably true or provably false), then the set of degrees of belief consists of only two statements: "true" and "false". In general, for 2-valued logic, the set of degrees of belief forms a *Boolean algebra*. For 2-valued logic, it is not necessary to describe *all* operations: it is sufficient, e.g., to describe $\&$ and $\neg$, then $\vee$ and $\rightarrow$ can be expressed in terms of these two connectives: $x \vee y$ is equivalent to $\neg((\neg x)\&(\neg y))$, and $x \rightarrow y$ is equivalent to $y \vee (\neg x)$.

Alternatively, one can choose $\&$ as the only basic operation, define the implication $a \rightarrow b$ as $\sup\{z : x\&z \leq y\}$, and negation $\neg x$ as $x \rightarrow \perp$, where $\perp$ is the bottom element of the Boolean algebra.

A similar idea can be applied to *linear logic*. The resulting algebraic structure is described, e.g., in [7].

Comment. To clearly indicate the difference between operations of linear logic and operations of traditional logic, different symbols are used in linear logic. In this paper, following [7], we will use $\sim x$ for negation in linear logic, $\multimap$ for linear implication, $\star$ for "both", and $\wedge$ for the regular "and" (for which A"and"A is equivalent to A).

This regular "and" is simply a lattice operation for the order $<$ defined as follows: $A < B$ iff $A\&B$ is equivalent to A. So, instead of saying that we have *two* operations, we can say that we have a single operation $\star$ and an order $<$.

It is natural to assume that $A\&B$ means the same as $B\&A$, and that $A\&(B\&C)$ means the same as $(A\&B)\&C$. Therefore, the operation $\star$ is commutative and associative. In mathematical terms, this means that the set X of degrees of belief with an operation $\star$ forms a *commutative semigroup*.

The operations $\star$ and $<$ must also be consistent. As a result, we arrive at the following formalism (described in [7]).

4 An Algebraic Approach to Linear Logic: Formalism

Definition 1. [7] *A lattice ordered commutative semigroup $(X, <, \star, \mathbf{1})$ with unit $\mathbf{1}$ is called a quantale if the following two conditions hold:*

- *$(X, <)$ is a complete lattice;*
- *for every $x \in X$, and for every set $S \subseteq X$,*

$$x \star \bigvee_{s \in S} s = \bigvee_{s \in S} (x \star s).$$

Based on "and" ($\star$), we can define implication and negation in a way that is similar to 2-valued logic:

Definition 2. [7] *Let $(X, <, \star, \mathbf{1})$ be a quantale. Then:*

- *the value false $\bot$ is defined as the smallest element of the lattice $(X, <)$;*
- *linear implication $x \multimap y$ is defined as*

$$x \multimap y = \bigvee \{z : x \star z \leq y\};$$

- *the value true $\top$ is defined as $\bot \multimap \bot$;*
- *negation $\sim x$ as $x \multimap \bot$.*

The operation "x is absolutely true" ($!x$) is new; it cannot be described directly in terms of the already defined operations. To describe !, we must use its relationship with the previous operations:

Definition 3. [7] *By an linear logic, we mean a tuple*

$$(X, <, \wedge, \vee, \bot, \top, \multimap, \star, \sim, !, \mathbf{1}),$$

where:

- $(X, <, \star, \mathbf{1})$ *is a quantale, with lattice operations* $\wedge$ *and* $\vee$*;*
- $\bot$, $\top$, $\multimap$, *and* $\sim$ *are defined as in Definition 2; and*
- $! : X \to X$ *is a function that satisfies the following four conditions:*
  (i) *for every* $x \in X$, $!x \le x$*;*
  (ii) *for every* $x, y \in X$, *if* $!y \le x$, *then* $!y \le !x$*;*
  (iii) $1 = !\top$*;*
  (iv) *for every* $x, y \in X$, $!x \star !y = !(x \wedge y)$.

Comments.

- These four conditions naturally follow from the intended meaning of !: e.g., (i) means that our degree of belief that x is ***absolutely*** true cannot exceed the degree of belief in x, etc.
- The formalism described above is called *intuitionistic linear logic*, because for this formalism (similarly to the so-called intuitionistic logic), double negation $\sim (\sim x))$ of a statement x is not necessarily equivalent to the original statement x. The particular case for which $\sim\sim x = x$ is called *classical linear logic.*
- The author of [7] complains that so far, there are few applications of this algebraic formalism. We are now going to show that traditional fuzzy logic operations can be represented as a particular case of this formalism, and thus, numerous applications of fuzzy logic can be viewed as applications of linear logic.

5 Examples Related to Fuzzy Logic

In fuzzy logic, the set of degrees of belief X is an interval $[0, 1]$, with the natural order $<$. Possible "and" operations $\star$ correspond to *t-norms.* The most widely used t-norms are the following (see, e.g., [3, 5]):

- *idempotent* operation $x \star y = \min(x, y)$;
- *strict Archimedean* operations; the most widely used of such operations is $x \star y = x \cdot y$; an arbitrary strict Archimedean operation can be described by a formula $x \star y = \varphi^{-1}(\varphi(x) \cdot \varphi(y))$ for some strictly monotonic function $\varphi : [0, 1] \to [0, 1]$.

- *non-strict Archimedean* operations; the most widely used of such operations is $x \star y = \max(x+y-1, 0)$; an arbitrary strict Archimedean operation can be described by a formula $x \star y = \varphi^{-1}(\max\{\varphi(x)+\varphi(y)-1, 0\})$ for some strictly monotonic function $\varphi : [0,1] \to [0,1]$.

In the following, we will specify linear logics corresponding to each of the above t-norms.

The results are that for idempotent operation $\star$, the corresponding operation ! is not uniquely defined, whereas for non-idempotent t-norms the operation ! is uniquely defined.

6 Linear Logic Operations in Fuzzy Logic Case

6.1 Idempotent "And"-Operation

The case when $\star$ is an idempotent operation, i.e., when $x \star y = \min(x,y)$, is *not* a typical case of linear logic, because in this case, the two "and" operations simply coincide. However, we will consider this case, because min is one of the most important and most frequent operations in fuzzy logic applications.

For this case, linear logic is not uniquely determined. All possible linear logics are described by the following proposition:

PROPOSITION 1.

- *If in a linear logic,* $X = [0,1]$, $<$ *is a standard order, and* $x \star y = \min(x,y)$, *then:*
  - $\perp = 0$; $\top = 1$;
  - $x \multimap y = 1$ *if* $x \le y$ *and* $x \multimap y = y$ *if* $x > y$;
  - $\sim x = 1$ *if* $x = 0$, *and* $\sim x = 0$ *for* $x > 0$;
  - *there exists a subset* $F \subset [0,1]$ *such that* $!x = \sup\{f \in F : f \le x\}$.
- *Vice versa, for every set* $F \subseteq [0,1]$, *the tuple*

$$(X = [0,1], <, \wedge = \min, \vee = \max, \perp, \top, \multimap, \star, \sim, !, \mathbf{1})$$

with the above-defined $\perp = 0$, $\top = 1$, $\multimap$, $\sim$, *and* !, *is a linear logic.*

Comments.

- For reader's convenience, all the proofs are placed in the last section.
- Since, as we have mentioned, this is not a typical case of linear logic, $!x$ does not necessarily mean "x is absolutely true". It rather means that we choose some set F of *approximating* degrees of belief, and that $!x$ is an approximation from F; for example:

- If $F = [0,1]$, then $!x = x$.
- If $F = \{0,1\}$, then $!1 = 1$ and $!x = 0$ for $x < 1$.
- If for some n, $F = \{0, 1/n, 2/n, \ldots, 1 - 1/n, 1\}$, then $!x = \lfloor nx \rfloor / n$.

6.2 Strict Archimedean "And"-Operations

PROPOSITION 2. *If in a linear logic, $X = [0,1]$, $<$ is a standard order, and $x \star y = x \cdot y$, then:*

- $\perp = 0$; $\top = 1$;
- $x \multimap y = \min(1, y/x)$;
- $\sim x = 1$ *if* $x = 0$, *and* $\sim x = 0$ *for* $x > 0$;
- $!x = 1$ *if* $x = 1$, *and* $!x = 0$ *when* $x < 1$.

PROPOSITION 3. *If in a linear logic, $X = [0,1]$, $<$ is a standard order, and $\star$ is a generic strict Archimedean operation $x \star y = \varphi^{-1}(\varphi(x) \cdot \varphi(y))$, then:*

- $\perp = 0$; $\top = 1$;
- $x \multimap y = \min(1, \varphi^{-1}(\varphi(y)/\varphi(x))$;
- $\sim x = 1$ *if* $x = 0$, *and* $\sim x = 0$ *for* $x > 0$;
- $!x = 1$ *if* $x = 1$, *and* $!x = 0$ *when* $x < 1$.

6.3 Non-Strict Archimedean "And"-Operations

PROPOSITION 4. *If in a linear logic, $X = [0,1]$, $<$ is a standard order, and $x \star y = \min(x + y - 1, 0)$, then:*

- $\perp = 0$; $\top = 1$;
- $x \multimap y = \min(1, 1 + y - x)$;
- $\sim x = 1 - x$;
- $!x = 1$ *if* $x = 1$, *and* $!x = 0$ *when* $x < 1$.

Comment. In this case, $\sim\sim x = x$, so it is a classical linear logic.

PROPOSITION 5. *If in a linear logic, $X = [0,1]$, $<$ is a standard order, and $\star$ is an arbitrary non-strict Archimedean operation*
$x \star y = \varphi^{-1}(\max\{\varphi(x) + \varphi(y) - 1, 0\})$, then:

- $\perp = 0$; $\top = 1$;
- $x \multimap y = \min(1, \varphi^{-1}(1 + \varphi(y) - \varphi(x)))$;
- $\sim x = \varphi^{-1}(1 - \varphi(x))$;
- $!x = 1$ *if* $x = 1$, *and* $!x = 0$ *when* $x < 1$.

7 Proofs

7.1 Proof of Proposition 1

Since $<$ is a natural order on the interval $[0,1]$, we have $\perp = 0$ and $\vee = \sup$. Therefore, according to the definition of $\multimap$, we have

$$x \multimap y = \sup\{z : \min(x,z) \le y\}.$$

Then:

- If $x \le y$, then for every $z \in [0,1]$, we have $\min(x,z) \le x \le y$, and therefore, the desired supremum is equal to 1.
- Let us now consider the case when $x > y$. In this case, the inequality $\min(x,z) \le y$ is only possible when $z \le y$. The largest of such z is y. Hence, in this case, the desired supremum is equal to y.

For $x = y = 0$, we have $\min(z,0) = 0 \le 0$ for all z, so, $\top = 0 \multimap 0 = 1$.

For $x > 0$, we have $\sim x = x \multimap 0 = 0$.

Let us now describe possible operations !. For every !, we can define F as the range of !, i.e., as the set of all the values $!x$ for different $x \in [0,1]$. Then, according to part (i) of Definition 3, for every x, we have $!x \in F$. To complete the proof of the Proposition, it is sufficient to show that $!x$ is the largest element of F that is $\le x$. Indeed, let $f \in F$ and $f \le x$. Then, since $f \in F$, we have $f = !y$ for some y. According to part (ii) of Definition 3, we from $f = !y \le x$, it follows that $f = !y \le !x$, i.e., $!x$ is indeed the largest element of F that is $\le x$.

Vice versa, the fact that the above formula defines ! for every set F, follows from Theorem 8.18 [7]. Q.E.D.

7.2 Proof of Propositions 2 and 3

Let us first prove Proposition 2. Since $<$ is a natural order on the interval $[0,1]$, we have $\perp = 0$ and $\vee = \sup$. Therefore, according to the definition of $\multimap$, we have $x \multimap y = \sup\{z : x \cdot z \le y\}$. Then:

- If $x \le y$, then for every $z \in X = [0,1]$, we have $x \cdot z = y$ and therefore, the desired supremum is equal to 1.
- If $x > y$, then the inequality $x \cdot z \le y$ is only true for $z \le y/x$. So, the least upper bound of the set of all such z is equal to y/x.

Combining these two cases, we get the desired expression $\min(1, y/x)$.

The only case that is not covered by this definition is $x = y = 0$. In this case, $z \cdot x \le y$ for all z and therefore, $\{z : z \cdot x \le y\} = [0,1]$, and the supremum of this set is equal to 1. Hence, $\top = \perp \multimap \perp = 0 \multimap 0 = 1$.

Now, $\sim 0 = 0 \multimap 0 = 1$, and for $x > 0$, we have $\sim x = x \multimap 0 = \min(1, 0/x) = 0$.

Let us now find $!x$ for different x.

- Since we have proved that $\top = 1$, we can conclude from part (iii) of the definition of ! that $!1 = 1$.
- Let $x < 1$; then, by applying part (iv) of the definition of ! with $y = x$, we conclude that $(!x) \cdot (!x) = !x$, i.e., that $(!x)^2 = !x$. Therefore, $!x = 0$ or $!x = 1$. From part (i), it follows that $!x \leq x < 1$, so $!x = 1$ is impossible. Hence, in this case, $!x = 0$.

Proposition 2 is proven.

Using isomorphism φ between an arbitrary strict Archimedean operation and $x \cdot y$, we can easily extend this result to an arbitrary strict Archimedean operation $\star$, thus proving Proposition 3. Q.E.D.

7.3 Proof of Propositions 4 and 5

Let us first prove Proposition 4. Since $<$ is a natural order on the interval $[0,1]$, we have $\perp = 0$ and $\vee = \sup$. Therefore, according to the definition of $\multimap$, we have $x \multimap y = \sup\{z : \max(0, x+z-1) \leq y\}$. Since $0 \leq z$, the inequality $\max(0, x+z-1) \leq y$ is equivalent to $x+z-1 \leq y$, i.e., to $z \leq y+1-x$. Therefore, the linear implication $x \multimap y$ that is equal to the largest of the values $z \in [0,1]$ that satisfy this inequality, is equal to $\min(1, y+1-x)$.

In particular, for $x = y = 0$, we get $\top = 0 \multimap 0 = 1$.

Hence, $\sim x = x \multimap 0 = \min(1, 1-x) = 1-x$ for all $x \in X$.

To complete the proof of this Proposition, let us find $!x$. For $x = 1$, we have $!\top = !1 = 1$. For $x < 1$, we can apply (iv) with $y = x$, and conclude that $!x \star !x = \max(0, 2 \cdot !x - 1) = !x$. Hence, either $!x = 0$, or $!x = 2 \cdot !x - 1$. In the second case, $!x = 1$, but we have $!x \leq x < 1$. Hence, the second case is impossible, and for $x < 1$, we have $!x = 0$. Proposition 4 is proven.

Using isomorphism φ between an arbitrary non-strict Archimedean operation and $\max(x+y-1, 0)$, we can easily extend this result to an arbitrary non-strict Archimedean operation $\star$, thus proving Proposition 5. Q.E.D.

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