FUZZY α-ALMOST SUPRACOMPACTNESS IN FUZZY SUPRATOPOLOGICAL SPACES

A. Haydar Eş[(1)] - Doğan Çoker[(2)]

[(1)] *Department of Mathematics, Hacettepe University, Beytepe, 06532-Ankara / TURKEY*

[(2)] *Department of Mathematics Education, Hacettepe University, Beytepe, 06532-Ankara / TURKEY*

Abstract: The concept of fuzzy supratopological space was introduced by El-Monsef and E.Ramadan [1]. El-Monsef and E.Ramadan have introduced the concept of α-supracompactness in fuzzy supratopological spaces. S. Dang, A. Behera and S. Nanda [8] introduced the concepts of Hausdorfness and α-Lindelöfness in fuzzy supratopological spaces. In this paper, we introduce fuzzy α-almost supracompactness and we give some characterizations of fuzzy α-almost supracompactness. Also we investigate the image of fuzzy α-almost supracompact spaces under some types of functions.

Keywords: Fuzzy supratopological spaces; fuzzy α-supracompactness; fuzzy α-almost supracompactness; fuzzy almost supracontinuity; fuzzy weak supracontinuity; fuzzy α-S-supraclosedness.

1. Introduction

After the introduction of the fundamental concept of a fuzzy set defined by Zadeh in [16], fuzzy α-supracompactness is given and studied in fuzzy supratopological spaces by using the concept of α-shading [1]. In this paper we introduce and investigate the concept of fuzzy α-almost supracompactness. We also investigate the relations between this concept and another weak forms of fuzzy α-supracompactness in fuzzy supratopological spaces.

2. Preliminaries

The definitions of fuzzy sets, fuzzy topological spaces and other related concepts can be found in [16,6]; the definitions of α-shading, fuzzy α-almost compactness in [2,3,4,7,11,12,13]; the definitions of fuzzy regular spaces, fuzzy almost continuity and fuzzy weak continuity in [5,13]. The concepts of compactness, almost compactness, near compactness and RS-compactness for fuzzy sets were studied in [6,9,10,14], and some results in fuzzy supratopological spaces can be found in [8,15].

Correspondence to : Dr. A. Haydar Eş, Matematik Bölümü, Hacettepe Üniversitesi, Beytepe, 06532-ANKARA / TURKEY
[e-Mail: es@bool.mat.hun.edu.tr]

Definition 2.1. [1] A subclass $\tau \subseteq I^X$ is called a fuzzy supratopology on X if $0,1 \in \tau$ and τ is closed under arbitrary union. In this case (X,τ) is called a fuzzy supratopological space (a fsts for short), the members of τ are called fuzzy supraopen sets. A fuzzy set λ is fuzzy supraclosed iff its complement λ^c is fuzzy supraopen.

Theorem 2.2. [1] If λ, μ are fuzzy sets in the fsts X, then

(1) λ is fuzzy supraopen (resp. fuzzy supraclosed) iff $\lambda=\lambda^{os}$ (resp. $\lambda=\lambda^{-s}$).

(2) If $\lambda \le \mu$, then $\lambda^{os} \le \mu^{os}$ and $\lambda^{-s} \le \mu^{-s}$.

(3) $\lambda^{-s} \cap \mu^{-s} \ge (\lambda \cap \mu)^{-s}$.

(4) $\lambda^{os} \cup \mu^{os} \le (\lambda \cup \mu)^{os}$.

(5) $\lambda^{-s}= 1-(1-\lambda)^{os}$.

(6) $\lambda^{os}= 1-(1-\lambda)^{-s}$.

Definition 2.3. [1] A function $f : (X,\tau_1) \to (Y,\tau_2)$ between two fsts's is called fuzzy supracontinuous, if $f^{-1}(\tau_2)\subseteq\tau_1$.

Notice that, if $\alpha\in[0,1)$, any subcollection $\mathcal{U}\subseteq I^X$ satisfying the condition $\forall x\in X\ \exists \mu\in\mathcal{U}\ [\mu(x)>\alpha]$ is called an α-shading of X; while, if $\alpha\in(0,1]$, any subcollection $\mathcal{U}\subseteq I^X$ satisfying the condition $\forall x\in X$ $\exists \mu\in\mathcal{U}\ [\mu(x)\ge \alpha]$ is called an α^*-shading of X [11]. A subcollection of an α- (resp. α^*-)shading $\mathcal{U}$ of X which is also an α- (resp. α^*-) shading of X is known as an α- (resp. α^*-) subshading of $\mathcal{U}$.

Definition 2.4. Let (X,τ) be a fsts and $\alpha\in[0,1)$ (resp. $\alpha\in(0,1]$).

(a) A subcollection $\mathcal{U}\subseteq I^X$ is said to be a fuzzy supraopen α-shading (resp. α^*-shading) of X, if $\mathcal{U}$ is an α-shading and each member of $\mathcal{U}$ is a fuzzy supraopen set. A fsts is said to be fuzzy α- (resp. α^*-) supracompact, if every fuzzy supraopen α-shading (resp. α^*-shading) of X has a finite α-subshading (resp. α^*-subshading) [1].

(b) $\mathcal{U}\subseteq I^X$ is called a proximate α-shading (resp. proximate α^*-shading), if for each $x\in X$ there exists $\mu\in\mathcal{U}$ such that $\mu^{-s}(x)>\alpha$ (resp. $\mu^{-s}(x)\ge\alpha$).

Theorem 2.5. [1] Let X, Y be fsts's and, let $f : X \to Y$ be a fuzzy supracontinuous surjection. If X is fuzzy α-supracompact, then so is Y.

Definition 2.6. [1] A collection $\mathcal{F} \subseteq I^X$ is said to be α-centered if for all $\mu_1, \mu_2, \ldots, \mu_n \in \mathcal{F}$, there exists $x_o \in X$ such that $\mu_k(x_o) \geq 1-\alpha$ for all $k=1,2,\ldots,n$.

3. Fuzzy α- and α^*-almost supracompact spaces

Throughout this section, the abbreviation 'f' will stand for the word 'fuzzy'.

Definition 3.1. Let (X,τ) be a fsts. (X,τ) is said to be fuzzy α ($0 \leq \alpha < 1$) (resp. α^* $(0 < \alpha \leq 1)$)-almost supracompact, if every fuzzy supraopen α (resp. α^*)-shading family $\mathcal{B} \subseteq \tau$ of X has a finite proximate α (resp. α^*)-subshading.

Definition 3.2. $\mathcal{U} \subseteq I^X$ is said to be α (resp. α^*)-almost centered, if for all finite subfamily $\{\mu_1, \ldots, \mu_n\}$ of $\mathcal{U}$, there exists $x \in X$ such that $\mu_k^{os}(x) \geq 1-\alpha$ (resp. $\mu_k^{os}(x) > 1-\alpha$) for all $k=1,2,\ldots,n$.

Theorem 3.3. Let (X,τ) be a fsts. Then the following statements are equivalent:

(1) (X,τ) is fuzzy α-almost supracompact (resp. fuzzy α^*-almost supracompact).

(2) For all family $\mathcal{F}$ of fuzzy supraclosed sets such that $\forall x \in X\ \exists \mu \in \mathcal{F}$ satisfying $\mu(x) < 1-\alpha$ (resp. $\mu(x) \leq 1-\alpha$), there exists a finite subfamily $\mathcal{F}_o$ of $\mathcal{F}$ such that $\forall x_o \in X\ \exists \mu \in \mathcal{F}_o$ satisfying $\mu^{os}(x_o) < 1-\alpha$ (resp. $\mu^{os}(x_o) \leq 1-\alpha$).

Proof. [$1 \Rightarrow 2$:] Let $\mathcal{F}$ be a collection of f. supraclosed sets such that $\forall x \in X\ \exists \mu \in \mathcal{F}$ such that $\mu(x) < 1-\alpha$. Then $\mathcal{U} = \{1-\mu : \mu \in \mathcal{F}\}$ is a f. supraopen α-shading of X. Since (X,τ) is f. α-almost supracompact, there exists a finite subfamily $\mathcal{F}_o$ of $\mathcal{F}$ such that $\forall x \in X\ \exists \mu \in \mathcal{F}_o$ satisfying $(1-\mu)^{-s}(x) > \alpha$, i.e. $1-\mu^{os}(x) > \alpha \Longrightarrow \mu^{os}(x) < 1-\alpha$.

[2 $\Rightarrow$ 1 :] Suppose that $\mathcal{U}$ is a f. supraopen α-shading of X. Then $\mathcal{F}=\{1-\mu:\mu\in\mathcal{U}\}$ is a collection of f. supraclosed sets in X such that $\forall x\in X\ \exists\,\mu\in\mathcal{U}$ such that $(1-\mu)(x)=1-\mu(x)<1-\alpha$. Hence, by (ii), there exists a finite subfamily $\mathcal{U}_o$ of $\mathcal{U}$ such that $\forall x_o\in X\ \exists\,\mu\in\mathcal{U}_o$ such that $(1-\mu)^{os}(x_o)<1-\alpha$, i.e. $1-\mu^{-s}(x_o)<1-\alpha \Longrightarrow \alpha<\mu^{-s}(x_o)$, as required.

Hence (X,τ) is f. α-almost supracompact. ∎

Theorem 3.4. A fsts X is fuzzy α-almost supracompact iff for every α-almost centered (resp. α^*-almost centered) collection $\mathcal{F}$ of fuzzy supraclosed sets in X, there exists $x\in X$ such that $\mu(x)\geq 1-\alpha$ (resp. $\mu(x)>1-\alpha$) for all $\mu\in\mathcal{F}$.

Proof. [$\Rightarrow$:] Suppose that $\mathcal{F}$ is an α-almost centered collection of f. supraclosed sets in X such that $\forall x\in X\ \exists\,\mu\in\mathcal{F}$ such that $\mu(x)<1-\alpha$. Then the collection $\mathcal{B}=\{1-\mu:\mu\in\mathcal{F}\}$ is a f. supraopen α-shading of X. Since X is f. α-almost supracompact, there exists finitely many elements $\mu_1,\mu_2,\dots,\mu_n\in\mathcal{F}$ such that $\forall x\in X\ \exists\,\mu_i$ $(i=1,2,\dots,n)$ such that $(1-\mu_i)^{-s}(x)>\alpha$, i.e. $\mu_i^{os}(x)<1-\alpha$, which contradicts the fact that $\mathcal{F}$ is α-almost centered.

[$\Leftarrow$:] This can be proved in a similar fashion. ∎

It is easy to see that if (X,τ) is f. α-supracompact (resp. α^*-supracompact), then (X,τ) is also f. α-almost supracompact (resp. α^*-almost supracompact). But the converse is not true as shown by the following counter-example:

Example 3.5. Let τ be the f. supratopology on $X=\{1,2,3,\dots\}$ generated by the base $\{\,1,\mu_n : n\in\mathbf{N}^+\,\}$, where

$$\mu_n(m)=\begin{cases}0.4, & \text{if } m\neq n\\ 0.5, & \text{if } m=n\end{cases}$$

If we let $\alpha=0.45$, then (X,τ) is f. α-almost supracompact (resp. α^*-almost supracompact), but not f. α-supracompact (resp. α^*-supracompact).

Theorem 3.6. Let (X,τ_1) , (Y,τ_2) be two fsts's and $f : X \to Y$ a fuzzy supracontinuous surjection. If f is fuzzy α-almost supracompact, then so is Y.

Proof. Straightforward. ∎

Definition 3.7. A function $f : (X,\tau_1) \to (Y,\tau_2)$ from a fsts X to another fsts Y is called fuzzy almost supracontinuous iff $f^{-1}((\mu^{-s})^{os}) \in \tau_1$ for each $\mu \in \tau_2$.

Theorem 3.8. Let (X,τ_1), (Y,τ_2) be two fsts's and $f : X \to Y$ be a fuzzy almost supracontinuous surjection. If X is fuzzy α-almost supracompact, then so is Y.

Proof. Let $\mathcal{U}$ be a f. supraopen α-shading of Y. Then the family $\mathcal{A}=\{f^{-1}((\mu^{-s})^{os}): \mu\in\mathcal{U}\}$ is a subfamily of τ_1 and is an α-shading of X. If we let $y\in Y$, then there exists $x\in X$ with $f(x)=y$, since f is surjective. Since X is f. α-almost supracompact, there exists a finite subfamily $\mathcal{U}_o$ of $\mathcal{U}$ such that $\forall x\in X$ $\exists \mu\in\mathcal{U}_o$ satisfying $\alpha<(f^{-1}((\mu^{-s})^{os}))^{-s}(x)$. Therefore

$$\alpha < (f^{-1}((\mu^{-s})^{os}))^{-s}(x) \le (f^{-1}(\mu^{-s}))^{-s}(x) = f^{-1}(\mu^{-s})(x)$$

$$= \mu^{-s}(f(x)) = \mu^{-s}(y) \implies \alpha < \mu^{-s}(y),$$

follows. Hence Y is f. α-almost supracompact, too. ∎

Definition 3.9. Let (X,τ_1), (Y,τ_2) be two fsts's and $f : X \to Y$ a function. f is called fuzzy weakly supracontinuous iff $f^{-1}(\mu) \le (f^{-1}(\mu^{-s}))^{os}$ for each $\mu\in\tau_2$.

Theorem 3.10. Let $f:(X,\tau_1) \to (Y,\tau_2)$ be a fuzzy weakly supracontinuous surjection. If (X,τ_1) is fuzzy α-supracompact, then (Y,τ_2) is fuzzy α-almost supracompact.

Proof. Let $\mathcal{U}$ be a f. supraopen α-shading of Y. Since f is fuzzy weakly supracontinuous, we have $f^{-1}(\mu)\le(f^{-1}(\mu^{-s}))^{os}$ for each $\mu\in\tau_2$. Then $\mathcal{A}=\{(f^{-1}(\mu^{-s}))^{os}:\mu\in\mathcal{U}\}$ is a f. supraopen α-shading of X . Let $y\in Y$. Then $\exists x\in X$ such that $f(x)=y$ by the surjectivity of f. Since X is f. α-supracompact, there exists a finite subfamily $\mathcal{U}_o$ of

$\mathcal{U}$ such that $\forall x\in X\ \exists \mu\in\mathcal{U}_o$ satisfying $(f^{-1}(\mu^{-s}))^{os}(x)>\alpha$. Hence $\mu^{-s}(y)=\mu^{-s}(f(x))=f^{-1}(\mu^{-s})(x)\geq(f^{-1}(\mu^{-s}))^{os}(x)>\alpha \implies \mu^{-s}(y)>\alpha$. Hence Y is f. α-almost supracompact. ∎

Remark 3.11. The last three theorems are valid, if α is replaced by α^{*}.

Definition 3.12. A fuzzy set λ in a fsts X is called fuzzy regular supraopen (resp. fuzzy regular supraclosed) if $\lambda^{-s^{os}}=\lambda$ (resp. $\lambda^{os^{-s}}=\lambda$).

Theorem 3.13. A fsts X is fuzzy α-almost supracompact iff each fuzzy regular supraopen α-shading $\mathcal{U}$ of X has a finite proximate α-subhading.

Proof. Obvious. ∎

Definition 3.14. A fuzzy set λ of an fsts (X,τ) is called a fuzzy suprasemiopen (resp. fuzzy regular suprasemiopen) set, if there exists a fuzzy supraopen (resp. fuzzy regular supraopen) set μ of X such that $\mu \leq \lambda \leq \mu^{-s}$.

Definition 3.15. A fsts (X,τ) is called fuzzy α-S-supraclosed (resp. fuzzy α^{*}-S-supraclosed), if each f. suprasemiopen α-shading (resp. α^{*}-shading) $\mathcal{U}$ of X has a finite proximate α-shading (resp. a finite proximate α^{*}-shading) of X.

Remark 3.16. It is clear that each fuzzy $\alpha(\alpha^{*})$-S-supraclosed fsts is also fuzzy $\alpha(\alpha^{*})$-almost supracompact. But the converse is not true:

Example 3.17. Let $X=\mathbb{Z}$ and consider the fuzzy sets μ_n, ν_n on X defined by

$$\mu_i(j)=\begin{cases} 0.7, & \text{if } i=j \\ 0.4, & \text{if } i\neq j \end{cases} \quad \text{and} \quad \nu_i(j)=\begin{cases} 0.2, & \text{if } i=j \\ 0.45, & \text{if } i\neq j \end{cases}.$$

Now let τ denote the fuzzy supratopology generated by $\{\mu_n, \nu_n : n=4,5,6,\ldots\}$. on X [12]. If $\alpha=0.75$, then (X,τ) is f. α^*-almost supracompact, since each f. supraopen α^*-shading of X must include 1_X as a member. But, since $\mu_i^{-s}=1-\upsilon_i$ $(i\in\mathbb{Z})$, $\{1-\upsilon_i : i\in\mathbb{Z}\}$ is a f. suprasemiopen α^*-shading of X and it has no finite proximate α^*-subshading; in other words, X is not f. α^*-S-supraclosed.

Theorem 3.18. A fsts X is fuzzy α-S-supraclosed iff every fuzzy regular supraclosed α-shading has a finite α-subshading.

Proof. Since a fuzzy regular supraclosed set is f. suprasemiopen and supraclosed, and the f. supraclosure of a f. suprasemiopen set is f. regular supraclosed, the proof is obvious. ∎

Theorem 3.19. A fsts X is fuzzy α-S-supraclosed iff each fuzzy regular suprasemiopen α-shading $\mathcal{U}$ of X has a finite proximate α-subshading.

Proof. Proceeding similarly as in Theorem 5.7 of [13], we obtain the result easily. ∎

Definition 3.20. The fsts (X,τ) is called fuzzy regular, if every fuzzy supraopen set λ of X can be written as the supremum of some fuzzy supraopen sets λ_i's of X such that $\lambda^{-s} \leq \lambda$ for each i.

Theorem 3.21. Let (X,τ) be a fuzzy regular fsts. If X is fuzzy α-S-supraclosed, then it is fuzzy α-supracompact, too.

Proof. It is analogous to the proof of Theorem 5.11 of [13]. ∎

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