FUZZY FLAT AND FAITHFULLY FLAT R-MODULES

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Abstract: In this paper by considering the notion of fuzzy exact sequence of fuzzy R-modules, the concepts of F-left exact and F-right exact functors are defined. It is shown that two functors $Hom_R(\theta_A, -)$ and $Hom_R(-, \eta_B)$ are not F-left exact, while in ordinary algebra they are left exact. Also it is seen that the functor $\theta_A \otimes_R -$ is F-right exact. Finally some equivalent conditions are proved, and on the basis of which the fuzzy flat and faithfully flat R-modules are defined.

Keywords: Fuzzy R-module, Fuzzy Tensor Product and Fuzzy Exact Sequence.

1. Preliminaries

Permouth [3] defined the tensor product of fuzzy R-modules. Zahedi and Ameri [7] defined and studied fuzzy exact sequence of fuzzy R-modules. In this paper R is a ring with identity and each module involved is an unitary R-module. All definitions and notations follow the presentations Pan [6], Permouth [3] and Zahedi and Ameri [7], unless otherwise stated.

Definition 1.1 [7]. The sequence

$$\cdots \to \mu_{n_{A_{n-1}}} \xrightarrow{\tilde{f}_{n-1}} \mu_{n_{A_n}} \xrightarrow{\tilde{f}_n} \mu_{n+1_{A_{n+1}}} \longrightarrow \cdots \tag{1}$$

of fuzzy R-module homomorphisms is called fuzzy exact when $im\tilde{f}_{n-1} = Ker\tilde{f}_n$, for all $n \in \mathcal{Z}$, where by $im\tilde{f}_{n-1}$ and $Ker\tilde{f}_n$ we mean $\mu|_{Kerf_n}$ and $\mu_n|_{Imf_{n-1}}$ respectively.

Example 1.2. Let $\tilde{f} : \mu_M \to \eta_N$ be a fuzzy homomorphism. Then the fuzzy sequence

$$\overline{0} \longrightarrow Ker\tilde{f} \xrightarrow{\tilde{i}} \mu_M \xrightarrow{\tilde{f}} \eta_N \xrightarrow{\tilde{\pi}} coker\tilde{f} \longrightarrow \tilde{0}$$

is exact, where i and π are the inclusion map and canonical epimorphism respectively and $coker\tilde{f} =_{,} \overline{\eta}_{(N/Imf)}$.

Definition 1.3 [7]. The fuzzy exact sequence

$$\overline{0} \to \mu_A \xrightarrow{\tilde{f}} \rho_B \xrightarrow{\tilde{g}} \eta_C \longrightarrow \overline{0} \tag{2}$$

is said to be a fuzzy short exact sequence.

2. Main Results

Definition 2.1. (i) The covariant (contravariant) functor $T(S) : R - fzmod \to fz - Ab$ is called F-left exact if for any fuzzy exact sequence $\overline{0} \to \mu'_{A'} \to \mu_A \to \mu''_{A''}$, $(\mu'_{A'} \to \mu_A \to \mu''_{A''} \to \overline{0})$, the sequence

$$0 \to T\mu'_{A'} \to T\mu_A \to T\mu''_{A''}$$

$$(0 \to S\mu''_{A''} \to S\mu_A \to S\mu'_{A'})$$

be exact, where $fz - Ab$ denotes the category of fuzzy Abelian groups.

(ii) The covariant functor $T : R -$ fzmod $\to fz - Ab$ is called F-right exact if for any fuzzy exact sequence $\mu'_{A'} \to \mu_A \to \mu''_{A''} \to \overline{0}$, the sequence

$$T\mu'_{A'} \to T\mu_A \to T\mu''_{A''} \to \overline{0}$$

be exact.

Theorem 2.2. $Hom(\theta_A, -)$ and $Hom(-, \eta_B)$ are respectively covariant and contravariant functors from R-fzmod to $fz-Ab$, where θ_A, η_B are fuzzy R-modules, and

$$Hom_R(\theta_A, -) : \Gamma_B \longmapsto Hom_R(\theta_A, \Gamma_B)$$

$$Hom_R(-, \eta_B) : \Gamma_A \longmapsto Hom_R(\Gamma_A, \eta_B).$$

Proof. By identifying the ordinary group G with the fuzzy subgroup χ_G, the proof is simple.

Remark 2.3. The following examples show that the covariant functor $Hom_R(\theta_M, -)$ and the contravariant functor $Hom_R(-, \eta_N)$ are not F-left exact, whenever the functors $Hom_R(M, -)$ and $Hom_R(-, N)$ are left exact in ordinary algebra. However Theorems 3.17 and 3.18 of [7] give necessary and sufficient conditions for F-exactness of the functors $Hom_R(\theta_M, -), Hom_R(-, \eta_N)$.

Example 2.4 [7]. Let $M \neq < 0 >$ be an R-module. Define the fuzzy modules μ, η, ρ and θ as follows:

$$\mu = \chi_{\{0\}} \ , \ \eta = \chi_M \ , \ \theta(x) = \begin{cases} 1 & if \ \ x = 0 \\ 1/3 & otherwise \end{cases} \ , \ \rho(x) = \begin{cases} 1 & if \ \ x = 0 \\ 1/2 & otherwise \end{cases}$$

Now if f is the identity map and g is the zero map on M, then it is easy to see that the fuzzy sequence $\overline{0} \to \mu_M \xrightarrow{\tilde{f}} \rho_M \xrightarrow{\tilde{g}} \eta_M$ is exact, but the sequence

$$0 \to Hom_R(\theta_M, \mu_M) \xrightarrow{\tilde{f}_*} Hom_R(\theta_M, \rho_M) \xrightarrow{\tilde{g}_*} Hom_R(\theta_M, \eta_M) \qquad (3)$$

is not exact, because $\tilde{1}_M \in Ker\tilde{g}_*$ and there is no $\tilde{\varphi} \in Hom_R(\theta_M, \mu_M)$ such that $\tilde{f}_*\tilde{\varphi} = \tilde{1}_M$.

Example 2.5 [7]. Let $M \neq < 0 >$ be an R-module. Define the fuzzy modules

μ, η, ρ and θ as follows:

$$\mu = \rho = \chi_{\{0\}} \ , \ \eta = \chi_M \ , \ \theta(x) = \begin{cases} 1 & if \ \ x = 0 \\ 1/3 & otherwise. \end{cases}$$

Now if f is the zero map and g is the identity map on M, then it is easy to check that the fuzzy sequence

$$\mu_M \xrightarrow{\tilde{f}} \rho_M \xrightarrow{\tilde{g}} \eta_M \longrightarrow \overline{0}$$

is exact. But the sequence

$$0 \to Hom_R(\eta_M, \theta_M) \xrightarrow{\tilde{g}^*} Hom_R(\rho_M, \theta_M) \xrightarrow{\tilde{f}^*} Hom_R(\mu_M, \theta_M) \tag{4}$$

is not exact, because $Im\tilde{g}^* \neq Ker\tilde{f}^*$.

Notation: Let μ_M be a right and ν_N a left fuzzy R-modules. Then the tensor product of μ_M and ν_N is denoted by $(\mu \otimes \nu)_{M \otimes_R N}$.

Lemma 2.6. Let ν_N be an arbitrary fuzzy R-module. Then ν_N induces the following two covariant functors

(i)
$$\nu_N \otimes_R - : R - \text{fzmod} \longrightarrow fz - Ab$$
$$\mu_M \longmapsto (\nu \otimes \mu)_{N \otimes_R M}$$

(ii)
$$- \otimes_R \nu_N : R - \text{fzmod} \longrightarrow fz - Ab$$
$$\eta_M \longmapsto (\eta \otimes \nu)_{M \otimes_R N}$$

Proof. Easy.

Theorem 2.7. Let ν_N be an arbitrary fuzzy R-module. Then the functors $\nu_N \otimes_R -$ and $- \otimes_R \nu_N$ are F-right exact.

Proof. We prove the theorem for $\nu_N \otimes_R -$. The proof for $- \otimes_R \nu_N$ is similar. Let the fuzzy sequence

$$\mu'_{M'} \xrightarrow{\tilde{f}} \mu_M \xrightarrow{\tilde{g}} \mu''_{M''} \longrightarrow \overline{0} \tag{5}$$

be exact. We show that the sequence

$$(\nu \otimes \mu')_{N \otimes_R M'} \xrightarrow{\tilde{1} \otimes \tilde{f}} (\nu \otimes \mu)_{N \otimes_R M} \xrightarrow{\tilde{1} \otimes \tilde{g}} (\nu \otimes \mu'')_{N \otimes_R M''} \longrightarrow \overline{0} \tag{6}$$

is exact. By [3] $\tilde{1} \otimes \tilde{f}$ and $\tilde{1} \otimes \tilde{g}$ are fuzzy homomorphisms. Since $1 \otimes g$ is onto, we conclude that $\tilde{1} \otimes \tilde{g}$ is an epimorphism [4]. Hence it is enough to show that $Im(\tilde{1} \otimes \tilde{f}) = Ker(\tilde{1} \otimes \tilde{g})$. Since the sequence (5) is fuzzy exact, by [7], the sequence $M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is also exact. Therefore the sequence

$$N \otimes_R M' \xrightarrow{1\otimes f} N \otimes_R M \xrightarrow{1\otimes g} N \otimes_R M'' \longrightarrow 0$$

is exact. Thus $Im(1 \otimes f) = Ker(1 \otimes g)$. Consequently

$$(\nu \otimes \mu)|_{Im(1\otimes f)} = \nu \otimes \mu|_{Ker(1\otimes g)}$$

Hence (6) is a fuzzy exact sequence.

Definition 2.8. If ν_N is a fuzzy R-module such that the functor $\nu_N \otimes_R -$ be F-left exact, then ν_N is said to be fuzzy flat.

Theorem 2.9. An R-module N is flat if and only if every fuzzy R-module ν_N is fuzzy flat.

Proof. Let N be a flat R-module and the sequence

$$\overline{0} \longrightarrow \mu'_{M'} \xrightarrow{\tilde{f}} \mu_M \xrightarrow{\tilde{g}} \mu''_{M''}$$

be fuzzy exact. Since N is flat, we can conclude that the sequence

$$0 \longrightarrow N \otimes_R M' \xrightarrow{1\otimes f} N \otimes_R M \xrightarrow{1\otimes g} N \otimes_R M''$$

is also exact. Thus the fuzzy homomorphism

$$\tilde{1} \otimes \tilde{f} : (\nu \otimes \mu')_{N\otimes_R M'} \longrightarrow (\nu \otimes \mu)_{N\otimes_R M}$$

is a fuzzy monomorphism [4]. Moreover

$$(\nu \otimes \mu)|_{Im 1\otimes f} = (\nu \otimes \mu)|_{Ker 1\otimes g}.$$

Thus the sequence

$$\overline{0} \longrightarrow (\nu \otimes \mu')_{N\otimes_R M'} \xrightarrow{\tilde{1}\otimes\tilde{f}} (\nu \otimes \mu)_{N\otimes_R M} \xrightarrow{\tilde{1}\otimes\tilde{g}} (\nu \otimes \mu'')_{N\otimes_R M''}$$

is fuzzy exact. That is ν_N is fuzzy flat. The proof of the converse is obvious.

Theorem 2.10. Let ν_N be a fuzzy R-module. Then the following conditions are equivalent:

(i) ν_N is fuzzy flat.

(ii) If $\overline{0} \to \mu'_{M'} \to \mu_M \to \mu''_{M''} \to \overline{0}$ is fuzzy exact sequence of fuzzy R-modules, then the sequence

$$\overline{0} \longrightarrow (\nu \otimes \mu')_{N \otimes_R M'} \longrightarrow (\nu \otimes \mu)_{N \otimes_R M} \longrightarrow (\nu \otimes \mu'')_{N \otimes_R M''} \longrightarrow \overline{0}$$

is fuzzy short exact.

(iii) If $\tilde{f} : \mu'_{M'} \to \mu_M$ is a fuzzy R-module monomorphism, then

$$\tilde{1} \otimes \tilde{f}(\nu \otimes \mu')_{N \otimes_R M'} \longrightarrow (\nu \otimes \mu)_{N \otimes_R M}$$

is also fuzzy monomorphism.

(iv) If $\tilde{f} : \mu'_{M'} \to \mu_M$ is a fuzzy R-module monomorphism and $\mu_M, \mu'_{M'}$ are fuzzy finitely generated R-modules, then

$$\tilde{1} \otimes \tilde{f}(\nu \otimes \mu')_{N \otimes_R M'} \longrightarrow (\nu \otimes \mu)_{N \otimes_R M}$$

is a fuzzy monomorphism.

Proof. $(i) \to (ii)$ This follows from Theorem 2.7 and Definition 2.8.

$(ii) \to (iii)$ Since $\tilde{f}$ is a monomorphism, thus Example 1.2 shows that the sequence

$$\overline{0} \longrightarrow \mu'_{M'} \xrightarrow{\tilde{f}} \mu_M \xrightarrow{\tilde{\pi}} coker \tilde{f} \longrightarrow \overline{0}$$

is fuzzy short exact. So by hypothesis the sequence

$$\overline{0} \longrightarrow (\nu \otimes \mu')_{N \otimes_R M'} \xrightarrow{\tilde{1} \otimes \tilde{f}} (\nu \otimes \mu)_{N \otimes_R M} \xrightarrow{\tilde{1} \otimes \tilde{\pi}} (\nu \otimes \overline{\mu})_{N \otimes_R M/Imf} \longrightarrow \overline{0}$$

is exact. Hence $\tilde{1} \otimes \tilde{f}$ is a monomorphism.

$(iii) \rightarrow (iv)$ This is obvious.

$(iv) \rightarrow (i)$ Let $f : M' \rightarrow M$ be a monomorphism and M, M' be finitely generated. Then the fuzzy map $\tilde{f} : (\chi_0)_{M'} \rightarrow (\chi_0)_M$ is a fuzzy monomorphism. Thus by hypothesis and definition of finitely generated R-module [6] we have that

$$\tilde{1} \otimes \tilde{f} : (\nu \otimes (\chi_0))_{N \otimes_R M'} \longrightarrow (\nu \otimes \chi_0)_{N \otimes M}$$

is a fuzzy monomorphism. Therefore by Theorem 3.7 of [4] $1 \otimes f : N \otimes_R M' \longrightarrow N \otimes_R M$ is 1-1. So N is a flat R-module [1, Proposition 2.19]. Hence by Theorem 2.9 ν_N is fuzzy flat.

Theorem 2.11. For any fuzzy R-module ν_M the following conditions are equivalent:

(i) The fuzzy sequence

$$\overline{0} \longrightarrow \mu'_{N'} \xrightarrow{\tilde{f}} \mu_N \xrightarrow{\tilde{g}} \mu''_{N''} \longrightarrow \overline{0}$$

of fuzzy R-module is exact if and only if the sequence

$$\overline{0} \longrightarrow (\nu \otimes \mu')_{M \otimes_R N'} \xrightarrow{\tilde{1} \otimes \tilde{f}} (\nu \otimes \mu)_{M \otimes_R N} \xrightarrow{\tilde{1} \otimes \tilde{g}} (\nu \otimes \mu'')_{M \otimes_R N''} \longrightarrow \overline{0}$$

is fuzzy exact.

(ii) If ν_M is fuzzy flat, then for any fuzzy R-module $\mu_N, (\nu \otimes \mu)_{M \otimes_R N} = \overline{0}$, implies that $\nu_N = \overline{0}$.

(iii) ν_M is fuzzy flat and for any fuzzy R-module homomorphism $\tilde{f} : \mu'_{N'} \rightarrow \mu_N$, if the induced fuzzy map

$$\tilde{1} \otimes \tilde{f}(\nu \otimes \mu')_{M \otimes_R N'} \longrightarrow (\nu \otimes \mu)_{M \otimes_R N}$$

is the zero map, then $\tilde{f} = \tilde{0}$.

Proof. By considering Theorem 2 on page 24 of [2], the proof is obvious.

Definition 2.12. If the R-module ν_M, satisfies one of the conditions of Theorem 2.11, then ν_N is called fuzzy faithfully flat.

Corollary 2.13. ν_M is fuzzy faithfully flat if and only if M is faithfully flat.

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