ENDOMORPHISM OF FUZZY POWER SET BASED ON LOWER CUT SET OF FUZZY SETS

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ABSTRACT

In this paper, a new kind of endomorphisms of fuzzy power set F(X) are discussed, which is based on lower cut set of fuzzy sets and is generalization of decomposition theorem in [1].

Keywords: Fuzzy Sets, Lower cut sets, Endomorphism, Decomposition theorem.

1. Lower cut sets and Decomposition theorem[1]

Let $F(X)=\{A|A:X\to[0,1]$ is a mapping$\}$. For $A\in F(X),\lambda\in[0,1],A^{\dot{\lambda}}=\{x|x\in X,A(x)\leqslant\lambda\},A^{\lambda}=\{x|x\in X,A(x)<\lambda\}$. $A^{\lambda},A^{\dot{\lambda}}$ are called as λ-lower cut set and λ-strong lower cut set of fuzzy A respectively.

Let C be a subest of set X, $\lambda\in[0,1]$, λC is defined as a fuzzy set of X and.

$$(\lambda C)(x)=\begin{cases}\lambda, if\ x\in C\\ 1, if\ x\notin C\end{cases}$$

then, We have decomposition theorems as following:

Theorem 1.1 $A=\bigcap\limits_{\lambda\in[0,1]}\lambda A^{\lambda}$, *i.e.* $A(x)=\wedge\{\lambda|\lambda\in[0,1],A(x)\leqslant\lambda\}$

Theorem 1.2 $A=\bigcap\limits_{\lambda\in[0,1]}\lambda A^{\dot{\lambda}}$, *i.e.* $A(x)=\wedge\{\lambda|\lambda\in[0,1],A(x)<\lambda\}$

Theorem 1.3 Let $H:[0,1]\to\mathscr{P}(x)$. $\lambda\to H(\lambda)$ satisfy: $A^{\dot{\lambda}}\subseteq H(\lambda)\subseteq A^{\lambda}$, then

(1) $\lambda_1<\lambda_2\Rightarrow H(\lambda_1)\subseteq H(\lambda_2)$

(2) $A=\bigcap\limits_{\lambda\in[0,1]}\lambda H(\lambda)$, *i.e.* $A(x)=\wedge\{\lambda|\lambda\in[0,1],x\in H(\lambda)\}$

(3) $A^{\lambda}=\bigcap\limits_{\alpha>\lambda}H(\alpha)$. $A^{\dot{\lambda}}=\bigcup\limits_{\alpha<\lambda}H(\alpha)$

In this paper, based on lower cut set of fuzzy set A, a new kind of endomorphisms of F(X) are discussed, which is a generalization of decomposition theorem as above.

2. Endomorphism of F(X)

Definition 2.1 Let $\sigma: F(X)\rightarrow F(X)$ be a mapping, if (i) $\sigma(\bigvee_{t\in T}\lambda_t)=\bigvee_{t\in T}\sigma(\lambda_t)$, $\forall\ \lambda_t\in[0,1]$; (ii) $\sigma(\bigwedge_{t\in T}\lambda_t)=\bigwedge_{t\in T}\sigma(\lambda_t)$, $\forall\ \lambda_t\in[0,1]$; (iii) $\sigma(A)(x)=\sigma(A(x))$, for any $A\in F(X)$, $x\in X$, then σ is called as a endomorphism of F(X).

Remark: For any $\lambda\in[0,1]$, Let $\lambda(x)=\lambda$, $\forall\ x\in X$, then $\lambda\in F(X)$. Clearrly, we have

Proposition Let σ be a endomorphism of F(X), then

(1) $\sigma(\bigcup_{t\in T}A_t)=\bigcup_{t\in T}\sigma(A_t)$ (2) $\sigma(\bigcap_{t\in T}At)=\bigcap_{t\in T}\sigma(At)$

Lemma 1 Let function $g:[0,1]\rightarrow[0,1]$ is right continuous, then

$$\bigcap_{\lambda\in[0,1)}g(\lambda)A^{\dot{\lambda}}=\bigcap_{\lambda\in[0,1)}g(\lambda)A^{\lambda} \qquad (1)$$

Proof: when $A(x)=1$, we have $x\notin A^{\dot{\lambda}}$ and $x\notin A^{\lambda}$ for any $\lambda<1$, then $\bigcap_{\lambda\in[0,1)}g(\lambda)A^{\dot{\lambda}}=1=\bigcap_{\lambda\in[0,1)}g(\lambda)A^{\lambda}$

when $A(x)<1$, we have $(\bigcap_{\lambda\in[0,1)}g(\lambda)A^{\dot{\lambda}})(x)=\bigwedge_{A(x)<\lambda<1}g(\lambda)\leqslant g(A(x))$, it follows that $\bigwedge\{g(\lambda)\,|\,A(x)<\lambda\}=\bigwedge\{g(\lambda)\,|\,A(x)\leqslant\lambda\}$, *Hence* $\bigcap_{\lambda\in[0,1)}g(\lambda)A^{\dot{\lambda}}=\bigcap_{\lambda\in[0,1)}g(\lambda)A^{\lambda}$

Corollary 1 Let mapping $H: [0,1]\rightarrow[0,1]\rightarrow\mathscr{P}(X)$ *satisfy*: $A^{\dot{\lambda}}\subseteq H(\lambda)\subseteq A^{\lambda}$, $\forall\ \lambda\in[0,1]$, *then* $\bigcap_{\lambda\in[0,1)}g(\lambda)A^{\dot{\lambda}}=\bigcap_{\lambda\in[0,1)}g(\lambda)H(\lambda)=\bigcap_{\lambda\in[0,1)}g(\lambda)A^{\lambda}$

Corollary 2 If $g(1)=\max\{g(\lambda)\,|\,\lambda\in[0,1]\}$, then $\bigcap_{\lambda\in[0,1]}g(\lambda)A^{\dot{\lambda}}=\bigcap_{\lambda\in[0,1]}g(\lambda)H(\lambda)=\bigcap_{\lambda\in[0,1]}g(\lambda)A^{\lambda}$

Remark: Let $\sigma_g(A)=\bigcap_{\lambda\in[0,1]}g(\lambda)A^{\lambda}$, $\forall\ A\in F(X)$, if $g(\lambda)=\lambda$, *then* $\sigma_g(A)=A$, *so* $\sigma_g(A)=\bigcap_{\lambda\in[0,1]}g(\lambda)A^{\lambda}$ is a generalization of decomposition theorem, we shall prove that σ_g is a endomorphism of F(X) when g is a continuous function.

Theorem 2.1 If g is a continuous function, then

(1) $\sigma_g(\bigcap_{t\in T}A_t)=\bigcap_{t\in T}\sigma_g(A_t)$ (2) $\sigma_g(\bigcup_{t\in T}A_t)=\bigcup_{t\in T}\sigma_g(A_t)$

Proof: (1) $\sigma_g(\bigcap_{t\in T}A_t)=\bigcap_{\lambda\in[0,1]}g(\lambda)(\bigcap_{t\in T}A_t)^{\lambda}=(\bigcap_{\lambda\in[0,1)}g(\lambda)(\bigcap_{t\in T}A_t)^{\lambda})\cap(g(1)$

$(\bigcap_{t\in T} A_t)=(\bigcap_{\lambda\in[0,1)} g(\lambda)(\bigcap_{t\in T} A_t)^{\dot\lambda})\cap(g(1)X)=(\bigcap_{\lambda\in[0,1)} g(\lambda)(\bigcup_{t\in T} A_t^{\dot\lambda}))\cap g(1)=$
$(\bigcap_{\lambda\in[0,1)}\bigcap_{t\in T} g(\lambda)A_t^{\dot\lambda})\cap g(1)=(\bigcap_{t\in T}(\bigcap_{\lambda\in(0,1)} g(\lambda)A_t^{\dot\lambda}))\cap g(1)=(\bigcap_{t\in T}(\bigcap_{\lambda\in[0,1)} g(\lambda)$
$A_t^{\lambda}))\cap g(1)=\bigcap_{t\in T}(\bigcap_{\lambda\in[0,1]} g(\lambda)A_t^{\lambda})=\bigcap_{t\in T}\sigma_g(A_t)$

(2) $\sigma_g(\bigcup_{t\in T} A_t)=\bigwedge_{\lambda\in[0,1]}\{g(\lambda)\mid\bigvee_{t\in T} A_t(x)\leqslant\lambda]$. Clearly, $\forall\, t\in T$, *we have* $\bigwedge_{\lambda\in[0,1]}\{g(\lambda)\mid A_t(x)\leqslant\lambda]\leqslant\bigwedge_{\lambda\in[0,1]}\{g(\lambda)\mid\bigvee_{t\in T} A_t(x)\leqslant\lambda$, then

$$\bigvee_{t\in T}(\bigwedge_{\lambda\in[0,1]}\{g(\lambda))\mid A_t(x)\leqslant\lambda\})\leqslant\bigwedge_{\lambda\in[0,1]}\{g(\lambda)\mid\bigvee_{t\in T} A_t(x)\leqslant\lambda] \tag{2}$$

If "$<$" is true in (2), then there is a $\alpha\in(0,1)$ such that

$$\bigvee_{t\in T}(\bigwedge_{\lambda\in[0,1]}\{g(\lambda)\mid A_t(x)\leqslant\lambda]<\alpha<\bigwedge_{\lambda\in[0,1]}\{g(\lambda)\mid\bigvee_{t\in T} A_t(x)\leqslant\lambda]] \tag{3}$$

then, $\bigwedge_{\lambda\in[0,1]}\{g(\lambda)\mid A_t(x)\leqslant\lambda]<\alpha$, for each $t\in T$, it follows that $\forall\, t\in T$, $\exists\,\lambda_t\in[0,1]$ Such that $A_t(x)\leqslant\lambda_t$ and $g(\lambda_t)<\alpha$, then $\bigvee_{t\in T} A_t(x)\leqslant\bigvee_{t\in T}\lambda_t=\beta$. Let $\{\lambda_{t_n}\}\subseteq\{\lambda_t\mid t\in T]$ and $\lim_{n\to\infty}\lambda_{t_n}=\beta$, then $g(\beta)=\lim_{n\to\infty} g(\lambda_{t_n})\leqslant\alpha$ and $\bigwedge_{\lambda\in(0,1)}\{g(\lambda)\mid\bigvee_{t\in T} A_t(x)\leqslant\lambda]\leqslant g(\beta)\leqslant\alpha$, This contradicts with (3). Hence "$=$" is true in (2), i. e., $\sigma_g(\bigcup_{t\in T} A_t)=\bigcup_{t\in T}\sigma_g(A_t)$

Lemma 2 Let $g:[0,1]\to[0,1]$ be a continuous function, if $f(\alpha)=\bigwedge_{\lambda\geqslant\alpha} g(\lambda)$, $\forall\,\alpha\in I$, then

(i) f is a continuous and monotone function

(ii) $\sigma_g=\sigma_f$

(iii) $\forall\,\lambda\in I$, $\sigma_f(\lambda)=f(\lambda)$

Theorem 2. 2 Let σ is a endomorphism of F(X), then there enists an unique continuous function $g:[0,1]\to[0,1]$ such that $\sigma=\sigma_g$.

Proof: Let $g(\lambda)=\sigma(\lambda)$, $\forall\,\lambda\in I$, then one can easily prove that $\sigma=\sigma_g$ and g is unique.

Theorem 2. 3 Let g be continuous. then σ_g is surjective if and only if (i) g (1)=1, (ii) g(α)=0 for some one $\alpha\in[0,1]$

Proof: "$\Rightarrow$" Clearly

"$\Leftarrow$" Let f is a continuous and monotone increasing function from $[0,1]$ to $[0,1]$, and satisfy: $\sigma_g=\sigma_f$, we shall prove that σ_f is surjective.

Let $H(\lambda)=A^{f(\lambda)}$ for any $\lambda\in[0,1]$ and $\beta=\bigcap_{\lambda\in[0,1]}\lambda H(\lambda)$, then $B^{\dot\lambda}\subseteq H(\lambda)\subseteq B^{\lambda}$ and

$$\bigcap_{\lambda\in[0,1]} f(\lambda)B^{\dot\lambda}\subseteq\bigcap_{\lambda\in[0,1]} f(\lambda)H(\lambda)\subseteq\bigcap_{\lambda\in[0,1]} f(\lambda)B^{\lambda}=\sigma_f(B)$$

Since $g(1)=\bigwedge_{\lambda\geqslant 1} g(\lambda)=\sigma_g(1)=\sigma_f(1)=f(1)$, so $f(1)=1$ and it follows that $\sigma_f(B)=\bigcap_{\lambda\in[0,1]} f(\lambda)B^{\lambda}$ and $\sigma_f(B)=\bigcap_{\lambda\in[0,1]} f(\lambda)B^{f(\lambda)}$.

By $f(0)=\bigwedge_{\lambda>0} f(\lambda)=\sigma_f(0)=\sigma_g(0)=\bigwedge_{\lambda>0} g(\lambda)=0$, we have $\{f(\lambda)\mid\lambda\in[0,1]\}=[0,1]$. It follows that $\sigma_f(B)=\bigcap_{\lambda\in[0,1]} f(\lambda)A^{f(\lambda)}=\bigcap_{\lambda\in[0,1)} \lambda A^{\lambda}=A$, i. e. σ_f is surjective.

Theorem 2.4 Let g be a continuous function from [0,1] to [0,1], then g is a monmorphism if and only if g satisfies: $\lambda_1<\lambda_2\Rightarrow g(\lambda_1)<g(\lambda_2)$

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