ON DISCONTINUOUS TRIANGULAR NORMS

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Abstract

In this paper two new methods which generate new triangular-norms are defined and investigated. The first is the so called 'cut t-norms' and the second is the 'fibre-bundle t-norms' which is closely related the homomorphism of semigroups. Several results and examples will be given.

Keywords
triangular norm, homomorphism, semigroup, equivalence relation, additive generator function

1 Introduction

In this paper two new methods which generate new triangular-norm from an arbitrary triangular norm will be defined and investigated. As the result (the range of that two mappings) two classes of discontinuous t-norms are given birth. We need only the basic definition of triangular norms: A triangular norm (t-norm for short) is a function T from $[0,1]^2$ to $[0,1]$ being commutative, associative, nondecreasing in each place and $T(1,x) = x$ holds for all $x \in [0,1]$. A t-norm T is said to be continuous if it is continuous as a two-place function. A t-norm T is called Archimedean if $T(x,x) < x$ is true for all $x \in (0,1)$.

This work is organised as follows. The cut t-norms will be introduced in Section 2 and the fibre-bundle t-norms in Section 3.

2 Cut t-norms

Definition 1 Let T be any t-norm and $\alpha \in [0,1]$. Let define the α-cut of T as follows:

$$T_{[\alpha]}(x,y) := \begin{cases} T(x,y) & \text{if } T(x,y) > \alpha \text{ and } x,y \in (0,1) \\ T(x,y) & \text{if } x,y \notin (0,1) \\ 0 & \text{if } T(x,y) \leq \alpha \text{ and } x,y \in (0,1) \end{cases}, \tag{1}$$

Theorem 1 *For any t-norm T its α-cut $T_{[\alpha]}$ is a t-norm.*

3 Fibre-bundle t-norms

3.1 T-norms generated by pseudo-authomorphisms

Let Ω denote the set of disjoint interval systems of the unit interval i.e.:

$$\Omega := \{\omega = \{(a_k, b_k] \subset [0,1] \mid k \in \Theta,\ [a_k, b_k] \cap [a_l, b_l] = \emptyset \text{ if } k \neq l\}\}$$

Let $f : [0,1] \to [0,\infty]$ be a monotone function. Obviously such an f admits the following representation: There exists a countable set Θ, and disjoint subintervals $(a_k, b_k]$ $(k \in \Theta)$ of the unit interval such that f is constant on each $(a_k, b_k]$ and strictly monotone on $[0,1] \setminus \cup_{k\in\Theta}(a_k, b_k]$. Let us call the set of this intervals the constant support of f and denote by $Supp_c(f)$. More formally, let

$$Supp_c(f) = \omega$$

where $\omega = \{(a_k, b_k]\ k \in \Theta : a_k < b_k, (a_k, b_k] \subset [0,1],\ f$ is constant on each $(a_k, b_k]$ and strictly monotone on $[0,1] \setminus \cup_{k\in\Theta}(a_k, b_k]\}$

The following definition for the non-decreasing case are due to [2]. The definition for the non-increasing case is an easy generalisation of the other case. If f is non-decreasing then let

$$f^{[-1]}(y) = \begin{cases} 1 & \text{if } y \geq f(0) \\ \sup\{x : f(x) < y\} & \text{if } y \in [0, f(0)] \end{cases} \tag{2}$$

If f is non-increasing then let

$$f^{[-1]}(y) = \begin{cases} 0 & \text{if } y \geq f(0) \\ \inf\{x : f(x) > y\} & \text{if } y \in [0, f(0)] \end{cases} \tag{3}$$

where $f^{[-1]}$ is called the pseudoinverse of f. Obviously if f is an additive generator function then $f^{[-1]}$ coincides the usual pseudoinverse $f^{(-1)}$ of an additive generator function.

Definition 2 We call $\phi : [0,1] \to [0,1]$ pseudo-authomorphism of the unit interval if it is a non-decreasing continuous function with $\phi(0) = 0$ and $\phi(1) = 1$. Let denote the set of pseudo-authomorphisms by $Aut_{ps}[0,1]$.

Let define

$$T_\phi(x,y) = \begin{cases} \phi^{[-1]}(T(\phi(x), \phi(y))) & \text{if } x, y \in [0,1) \\ \min(x,y) & \text{if } \max(x,y) = 1 \end{cases} \tag{4}$$

Theorem 2 *T_ϕ is a t-norm for any t-norm T.*

3.2 T-norms generated by projections

Take an element ω from Ω. Let denote $\xi(\omega) = 1 - \sum_{k\in\Theta}(b_k - a_k)$.

We define the following functions: $pr_\omega : [0,1] \to [0,1]$

$$pr_\omega(x) = \begin{cases} \frac{a_k - \lambda(a_k)}{\xi(\omega)} & \text{if } x \in (a_k, b_k] \text{ for some } k \in \Theta \\ \frac{x - \lambda(x)}{\xi(\omega)} & \text{otherwise} \end{cases} \tag{5}$$

where $\lambda : [0,1] \to [0,1]$, $\lambda(x) = \sum_{j:\ b_j < x}(b_j - a_j)$. One can check easily that thus defined pr_ω is a pseudo-authomorphism of $[0,1]$. Such a pseudo-authomorphism will be called a projection.

Corollary 1 *$T_{pr_\omega}(x,y)$ is a t-norm for any t-norm T.*

Then T is called the factor t-norm of T_{pr_ω} and T_{pr_ω} will be called the fibre-bundle t-norm of T generated by the interval system ω. This name is based on the observation that all the points of $[a_k, b_k]$ behaves equilly (i.e. $T(x_1,y) = T(x_2,y)$ if $x_1, x_2 \in [a_k, b_k]$ and $y \in [0,1)$ due to the properties of pr.

More formally, let $xRy \Leftrightarrow x$ and y belong to the interval $[a_k, b_k]$ for some $k \in \Theta$. Obviously it is an equivalence relation and T is the 'factor t-norm' on the quotient set (generated by R).

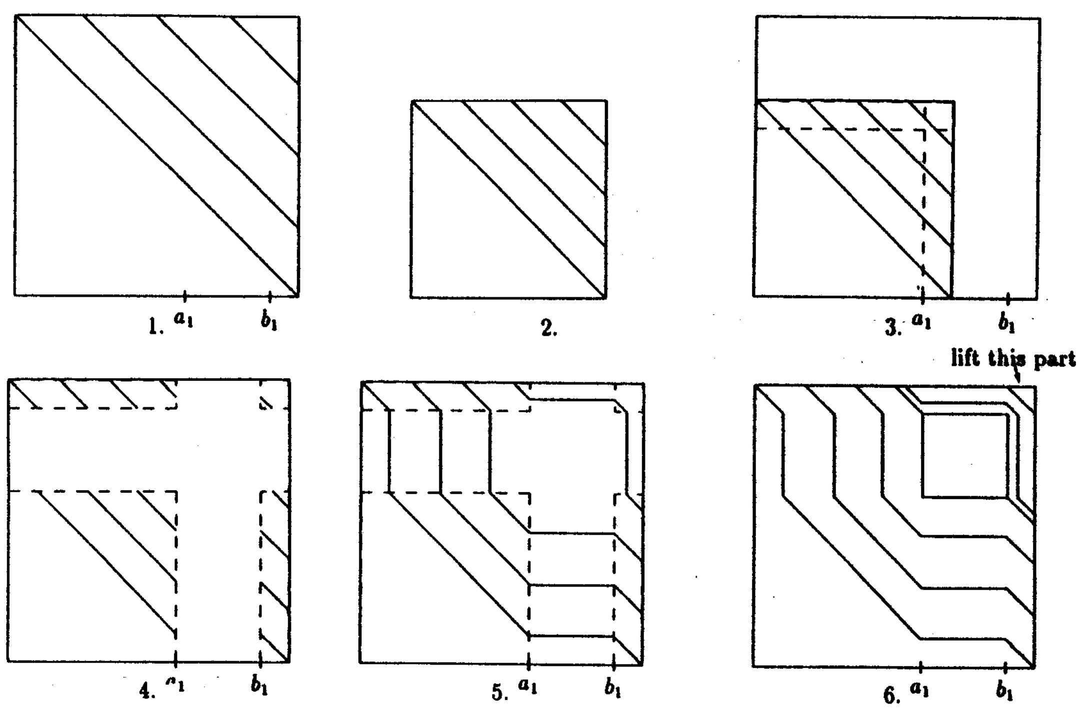
Another interpretation is given as follows: From the semigroup-theoretic point of view a t-norm which was generated with a pseudo-authomorphism (if we don't consider the boundary) is a homomorph image of its factor t-norm.

Theorem 3 *For a given pseudo-authomorphism ϕ there exists a projection pr and an authomorphism ψ such that*

$$T_\phi = (T_\psi)_{pr}$$

Example 1 Let $\omega = \{(a,b] | a \leq b\}$, T a t-norm. Consider the graph of T_{pr_ω}. Starting with an arbitrary t-norm T first we have to compress T like in the case of a 'summand' at the representation of the continuous t-norms [1]. Hence we have a t-norm on $[0,\xi(\omega)] \times [0,\xi(\omega)]$. Now we have to cut this graph along the lines $\{(x,y) : x \in [0,1], y = a\}$ and $\{(x,y) : y \in [0,1], x = a\}$, and then one has to shift the left and the upper parts to left and up respectively with $b-a$. Then one has to fill the lacks (i.e. $[a,b] \times [0,1]$ and $[0,1] \times [a,b]$) as defined and we have to cut our graph with a plane which is paralel to the domain, $[0,1] \times [0,1]$, at the high of a and have to lift the higher part of the graph with $b-a$. Finally, the boundary has to be redefined.

This process can be seen in the following picture when the factor t-norm is the Lukasiewicz t-norm. The lines in the unit square mean levels.



In general, when ω consist of more intervals then the process is almoast the same: one has to compress T with $\xi(\omega)$ and has to make as many cats and shifts as the cardinality of ω in order to obtain the lacks exactly at the intervals of the interval system ω. Then one has to make as many lifts as again the cardinality of ω in order to obtain the proper range and finally, the boundary has to be redefined.

Now we have a clear picture about the graph of a t-norm which was generated by a pseudo-authomorphism . In the light of Theorem 3 first we have to rescale the axes (with the authomorphism) then we have to change the graph in the way which was described in this example.

3.3 Weak additive generator function

Definition 3 Let $F : [0,1] \to [0,\infty]$ be a non-increasing continuous function with $F(1) = 0$. Then F is called a weak (additive) generator. Now we define the following two-place function:

$$T(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1 \\ F^{[-1]}(F(x) + F(y)) & \text{if } x,y \in [0,1) \end{cases} \tag{6}$$

Proposition 1 *The two-place function defined in* (4) *is a t-norm for any weak generator function* F.

Theorem 4 T^* *admits representation* (4) *if and only if there exists a continuous Archimedean t-norm* T *and a pseudo-authomorphism of the unit interval* ϕ *such that* $T^* = T_\phi$.

Theorem 5 *Suppose that* $0,1$ *is not in the closure of* $Supp_c(\phi)$. *Then* T *is Archimedean if and only if* T_ϕ *is Archimedean.*

3.4 Transformations of the minimum

The main idea of this paper was to find new t-norms in the form of

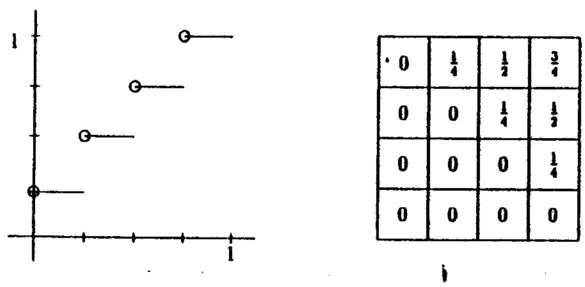
$$f^{[-1]} \circ T(f(x), f(y)) \text{ if } x,y \in [0,1). \tag{7}$$

For the sake of having the associativity property it was useful to suppose that $f^{[-1]}$ is a right-peudoinverse of f. In order not to have the value $T(f(x), f(y))$ out of the domain of $f^{[-1]}$ (which is the range of f) we supposed that f is continuous. But in same cases even continuity of f can be dropped out and we get to the definition of a pseudopseudo-authomorphism . We need the following condition:

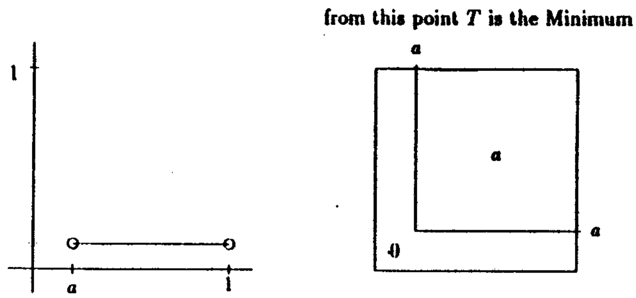
1.) Let f be 'closed under T' (i.e. the value of $T(f(x), f(y))$ should belong to the range of f for all x,y). Then formula (¶) defines a t-norm (of course T admits the boundary condition). (Notice that the pseudoinverse was defined even for the non-continuous case.) The investigation and characterisation of this problem is out of the scope of this paper. Only two examples are given.

The first can be considered as the discrete case of the Lukasiewicz t-norm. (This t-norm can be found in a forthcoming paper of Mirko Navara but it had been obtained there by using a different approach.)

The second example we are going to discuss is a wide family of t-norms. It is obvious that evaluating $T = \min$ we can use any pseudopseudo-authomorphism and (¶) defines a t-norm. Some examples are given here: The t-norms $T^{(n)} = \frac{E(\min(nx,ny))}{n}$ where n is an arbitrary natural number, and $E(x)$ is the integer part of the real x, and $T^{quadr}_{a,\varepsilon} = \begin{cases} 0 & \text{if } \min(x,y) \leq a \text{ and } \max(x,y) < 1 \\ \varepsilon & \text{if } \min(x,y) > a \text{ and } \max(x,y) < 1 \\ \min(x,y) & \text{if } \max(x,y) = 1 \end{cases}$
where $a \in [0,1]$, $\varepsilon \in [0,a]$ (introduced by Cappelmann, De Baets and Mesiar) is also within our framework (if $a = \varepsilon$): (Notice that $T^{quadr}_{0,0} = T_W$ the weakest t-norm.)



Discrete Lukasiewicz t-norm (here T is the Lukasiewicz t-norm



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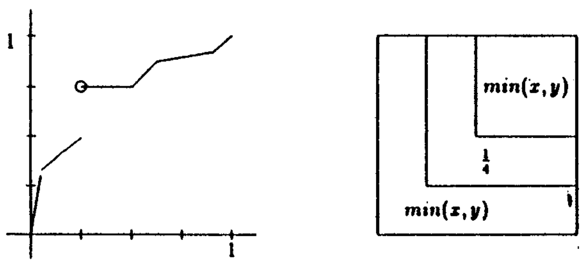
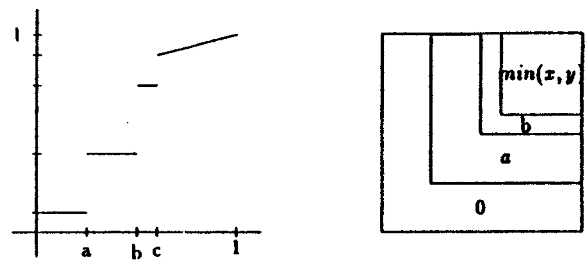
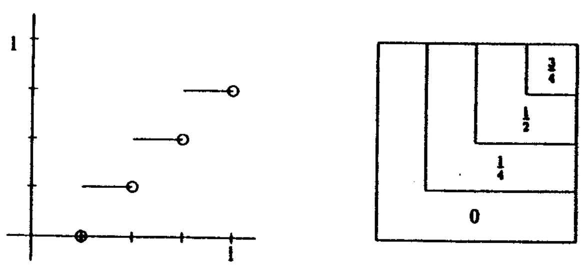
$T^{quadr}_{a,\varepsilon}$



Transform of the minimum t-norm



Discrete minimum t-norm



$T^{(4)}$

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