General and Specialised Solid Cutting Method for Fuzzy Rule Interpolation

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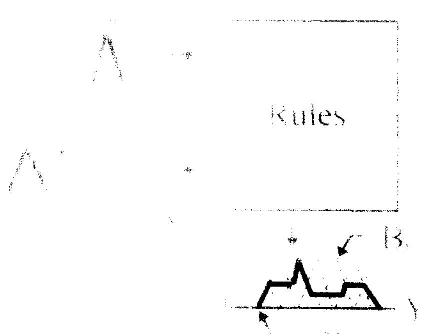
Abstract

In this article we will present a new fuzzy interpolation method. This ... compared to the existing methods, does not require convex and ... fuzzy sets in the rules, but can be applied for arbitrary type of fuzzy ... new method gives an interpretable conclusion in every case, ... the previously published methods.
... the article we will also show a specialized, simplified version of the ... method which uses three of the most wide spread set types in practice: ... the triangular, and the trapezoidal fuzzy sets. The difference ... the new and the former methods will be pointed out.

INTRODUCTION

...ntages of fuzzy controllers in practice, and the growing number of ... are well known. The basic job of fuzzy controllers is to transform the ... into membership degrees, then to generate conclusion fuzzy set and, ... needs, to deffuzify the conclusion [1].



The Mamdani-controller generates conclusion fuzzy set B' based on (n= number of inputs) inputs A_1'-A_n' (fig.1), knowing $A_{i,j}$ defined on input universes X_1-X_n, and B_i defined on output universe Y [2,3]. The number of rules is growing exponentially with the number of input variables and the number of fuzzy sets defined on the base sets. The total number of rules is: $r=\prod_{i=1}^{n} m_i$, where m_i are the numbers of sets (terms) of the i-th universe[4].

... of input universes is determined by the context of the system. The ... sets of the Mamdani-controller and of similar other type controllers is

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determined, because the intersection of the given terms of a universe and the observation can not be empty. In the practice, even stricter conditions apply: the input terms $A_{i,j}$ should be located densely. This means that the union of the α-cuts of the input fuzzy sets (e.g. for $\alpha = 0.5$) should be equal with the base set. The result of this "rule of thumb" is that even in case of a small system, the number of rules is increasing considerably. Having a large number of rules arises a lot of problems both in respect to calculation time and to storage space. There are different solutions for this problem in the literature [4,5,6,7].

The basic idea of these solutions is to generate the conclusion using only a small number of known sets defined on the input universes with a corresponding inference method. The application of such methods has an importance where the conclusion should be obtained from a small amount of knowledge identified before. So, knowing antecedents $A_{1..m}$ on X, and consequent $B_{1..m}$ on Y, relations $B_j = F^F{}_j\,(A_j)$ (j=1..m) are already known as fuzzy sets (here F^F denotes a relation. It is not a mathematical function, but a mapping from input (antecedent) fuzzy sets to consequent fuzzy sets). In case of observation A' clonclusion B' can be deduced by known relations between sets A_j and A_{j+1} sorrounding A' and their known consequents B_j and B_{j+1}. Therefore relation $B' = F^F(A')$ can be given by some weighted combination of relations $F^F{}_j$ and $F^F{}_{j+1}$. So these methods give conclusion $B' = B_j$ for different fuzzy sets $A' = A_j$ and generate conclusions for arbitrary observations between A_j and A_{j+1}.

The former interpolation methods induce several problems [8]:

- They can be applied only for convex and normal sets.
- They are not even interpetable for arbitrary convex and normal observation fuzzy sets, namely, ordering must hold ($A_j \prec A' \prec A_{j+1}$, $B_j \prec B_{j+1}$ where the observation set is A')
- The method does not give a directly interpretable conclusion fuzzy set in every case ("loops" in the membership functions).
- Using trapezoidalal, triangular or crisp sets, the shape is not preserved for the conclusion. It means that in case of fuzzy triangular or trapeziodal terms calculation by the three or four characteristic points e.g. by linear interpolation is not always sufficient as it gives only a rough approximation (except if some rather strict conditions apply). This is an important problem because of the computational complexity aspect.

In this article we will present a new method that can be applied on arbitrary shaped fuzzy terms and that always results in directly "acceptable" sets, further on eliminate all the mentioned problems. To show the essential new points in this method, we will classify the former interpolation methods by their key idea. Since we use only the terms flanking the observation let us use the following denotations: X is the input universe, A_1 and A_2 are the antecedents defined on X, Y is the output universe, B_1 and B_2 are the consequents defined on Y, A' is the observation and B' is the conclusion that has to be generated from A', A_1, A_2, B_1 and B_2.

The first class contains single term deduction methods. The conclusion (B') is generated from observation A' and A_1 a single rule, B_1 using some kind of General Modus Ponens (GMP see e.g. the Revision Principle [7]). These methods conclusion if the intersection of A' and A_1 is empty. The problem is that in this case the distance between A' and A_1 is not meaningful.

Methods in the second class, use at least two rules. The Mamdani-method and other reasoning methods alike use the degree of matching between observation and at least two antecedents by calculating a weighted average (see e.g.)[2]. This is a natural way of interpolation.

The third class applies approximation for the α-cuts and this can be used even if there is no formal matching. The linear interpolation method is the prototype of the methods in this class. These methods generate conclusion B' to observation A' using at least two rules ($A_1 \rightarrow B_1$, $A_2 \rightarrow B_2$) [4,5,6]. The basic idea of these methods is the following: If given are sequence of observations A_1, A', A_2 and a corresponding of sequence of conclusions B_1, B', B_2 where B' is unknown, B' is found by considering A'-s relative location in X and determining B' from B_1 and B_2 accordingly. If A' is not comparable with A_i ($A_1 \prec A' \prec A_2$) then this method can not be applied. However generalized interpolation and approximation methods (where polynomial or rational functions are used on the characteristic points of the α-cuts, eliminate the difficulty of orderedness condition [9].

For the same rules and observation some method gives a conclusion while some others do not. If a certain reasoning method gives a meaningless conclusion, it does not mean that there is no conclusion at all. The correctness of the conclusion depends on the semantic interpolation of $A_1..A_2$, $B_1..B_2$ that form the relevant part of the knowledge base, and on the semantics of A' and B'. The methods in the third class assume that there exists a "function" F from $F(X)$ to $F(Y)$ (where F denotes the fuzzy power set), with the following properties: $A'=F(A_1..A_2)$, $B'=F(B_1..B_2)$.

The method introduced in this article, goes back to the basic interpolation of fuzzy rules and searches for a conclusion similarly to the way of human thinking [10]. When we hear a new question, at first we summarize our knowledge that is closest to the topic of the question, and we try to find questions that approach the new question as close as possible and whose answers are known. Then based on the comparison of our factual knowledge and the new question we deduce an approximate answer. Following this method, at first, we look for a fitting A'' in the sequence of observation "between" known A_1 and A_2, which is closest to A', and we determinate the corresponding conclusion B'' in the sequence of conclusions "between" B_1 and B_2. Then by some inference method similar to the ones in the first class, by evaluating A', A'' and B'' the conclusion B' corresponding to A' will be found. The approach of this paper that is based on the previous considerations can be applied for arbitrary shape terms in the rules, and in the observation. Of course, if we use general type fuzzy terms, the operations with these sets will need much computational effort, because enough points of the sets should be taken into consideration to have a good enough approximation. Therefore, for practical applications we prepare a special method that can be applied for crisp, triangular and trapezoidal fuzzy sets (that can be described by 2,3 and 4 characteristic points resp.).

II. DEFINITIONS

1) $x_2 = F(x_1,p_1,p_2,c_1,c_2)$ is the revision function.

Let $x_2 = F(x_1,p_1,c_1,c_2)$ be that function, which results x_2 in such a way that a/b=c/d is true both if $x_1 \le p_1$ and $p_1 < x_1$ (see fig. 2).

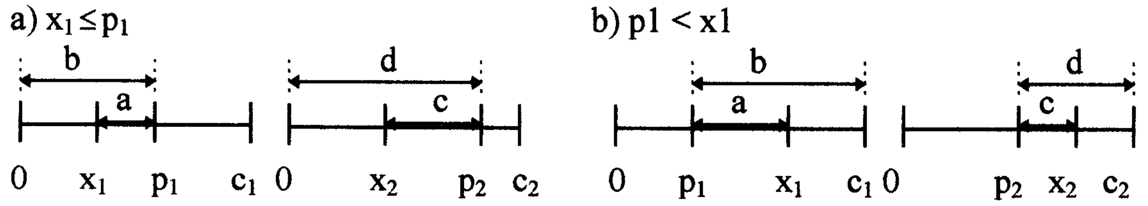


fig.2

$$x_2 = F(x_1, p_1, p_2, c_1, c_2) = \begin{cases} p_2; & \text{if } x_1 = p_1 \\ p_2 - a\dfrac{b \cdot c_2 - p_2}{b \cdot c_1 - p_1}; & \text{otherwise} \end{cases} \quad \text{where: } \begin{array}{l} a = p_1 - x_1; \\ b = \dfrac{1}{2}\left(1 - \operatorname{sgn}(a)\right); \end{array} \qquad (1)$$

2) cp(A) will be called the central point of fuzzy set A.

A $(\langle \forall x \in X,\ \mu_A(x) \rangle)$ fuzzy set is given on base set X. The center of the fuzzy set is:

$$cp(A) = \frac{\sup(A_\alpha) + \inf(A_\alpha)}{2}; \qquad \text{where } \alpha = \text{height}(A): \qquad (2)$$

(This is a generalization of the concept of the centre of the core.)

3.) $\text{suppnorm}_{SU}^{SL}(A)$ is the normalization of the support of fuzzy set A for given SU, SL.

Let A $(\langle \forall x \in X,\ \mu_A(x) \rangle)$ fuzzy set be given on base set X, then let $\text{suppnorm}_{SU}^{SL}(A)$ be such a fuzzy set SNA, whose support's minimum value is SL and maximum value is SU. Let the membership function of SNA= $\text{suppnorm}_{SU}^{SL}(A)$ be:

$$\mu_{SNA}(x) = \mu_A\left[(x - cp(A)) \cdot \left(\frac{a - cp(A)}{b - cp(A)} \right) + cp(A) \right]; \qquad (3)$$

Where: SU<cp(A)<SL;

If x<cp(A) then:
$a = \text{supp}_L(A) = \inf(\text{supp}(A))$;
$b = SL$;

If x>cp(A) then:
$a = \text{supp}_U(A) = \sup(\text{supp}(A))$;
$b = SU$;

III. GENERAL SOLID CUTTING METHOD

According to the Introduction, sets A" and B" are to be determined, so in the second step, B' can be determined using A', A" and B".

1) For determining A" and B" there are more conditions to be satisfied: A" should be as close to A' as possible. The closer is A" to A_j the more similar they are. The same "similarity relation" is required between B" and B_j. In an extreme case, when observation A' is identical with A_j (an already known antcedent) then A" the closest information to A' should be also identical with A_j. Similarly, B" should be identical with B_j, which implies that conclusions B' and B" are also the same. Therefore in the extreme case, when the observation is A_j, the conclusion should be B_j.

Let's define the crisp distance between two fuzzy sets with the distance of their centres [5]:

$$d(A_1,A_2) = d(cp(A_1), cp(A_2)). \tag{4}$$

To avoid the problem of abnormal membership function shapes for B' no other distance will be calculated, and all points of the membership function will be generated by this distance as a reference.

For simplicity, let us consider X and Y be normalized for the interval [0,100] i.e. Mx=max(supp(X))=My=max(supp(Y))=100, mx=min(supp(X))=my=min(supp(Y))=0.

Let us turn fuzzy sets A_1 and A_2 around their centers as it is shown in figure 3. The rotated curves (A_1 and A_2) are considered as the cross-sections of a geometric solid. Fuzzy set A" can be found between A_1 and A_2 as the cross section of this imaginary geometric solid. To get A", the closest fuzzy set to A', we have to cut the solid at the position of A', using the above introduced distance measure. Turning back the cross-section into its original position we will obtain A", a set that satisfies our conditions, namely that is equal to A_1 or A_2 in an extreme case. We determine B" similarly. Satisfying the conditions, the geometrical solid, that is created by turning B_1 and B_2, should be cut in such way that a/b is equal to c/d as it is shown in figures 3 and 5.

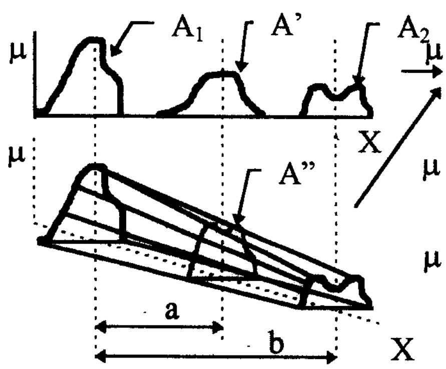


fig. 3

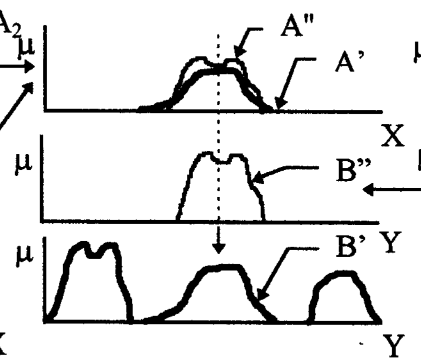
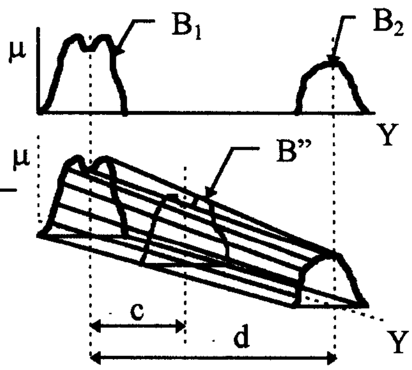


fig. 4



(a / b = c / d)

fig. 5

There are more possibilities to define the solid based on the rotated fuzzy sets. Therefore, it is possible to handle more than two sets A_i as one solid. In a simple case, when we consider only two fuzzy sets, the geometric solid based on A_1 and A_2 can be easily constructed as a linear translation surface with A_1 and A_2 as rule curves.

2) Set B' can be determined from A', A" and B" using a General Modus Ponens type method of the first class of the introduction's method, as it is shown in figure 4.

Unfortunately these methods can be applied only on certain restricted types of fuzzy sets. Therefore, to apply the generalized approach in every case, we have to use a different technique. Nevertheless this method uses the main idea of the in the „first class" reasoning approaches.

Our method satisfies the following conditions: B' should be as close to B" as A' is to A" and cp(A') should coincide with cp(A"), so let: cp(B') = cp(B"). The method consists of four steps:

a) Determination of the support of B' using A', A" and B".

$$\mathrm{supp}_L(B') = F(\mathrm{supp}_L(A'), \mathrm{supp}_L(A''), \mathrm{supp}_L(B''), cp(A''), cp(B''));\qquad (5)$$

$$\mathrm{supp}_U(B') = cp(B') + F(\ \mathrm{supp}_U(A') - cp(A'), \mathrm{supp}_U(A'') - cp(A''), \mathrm{supp}_U(B'') - cp(B''), Mx - cp(A''), My - cp(B''));\qquad (6)$$

b) Normalization of fuzzy sets A', A", and B" to the same support, while their center does not change. The size of the common support is arbitrary, in this case let it be d.

$$SL=cp(A')-d/2;\quad SU=cp(A')+d/2;\quad SNA'=\mathrm{suppnorm}_{SU}^{SL}(A');\qquad (7)$$

$$SL=cp(A'')-d/2;\quad SU=cp(A'')+d/2;\quad SNA''=\mathrm{suppnorm}_{SU}^{SL}(A'');\qquad (8)$$

$$SL=cp(B'')-d/2;\quad SU=cp(B'')+d/2;\quad SNB''=\mathrm{suppnorm}_{SU}^{SL}(B'');\qquad (9)$$

c)Determination of set SNB' point by point, using function F.

$$x_1 = \mu_{SNA'}(\mathrm{supp}_L(SNA') + y);\qquad (10)$$

$$p_1 = \mu_{SNA''}(\mathrm{supp}_L(SNA'') + y);\qquad (11)$$

$$p_2 = \mu_{SNB''}(\mathrm{supp}_L(SNB'') + y);\qquad (12)$$

$$c_1 = 1;\ c_2 = 1;\qquad y \in [0,d];$$

$$\mu_{SNB'}(\mathrm{supp}_L(SNB') + y) = F(x_1, p_1, p_2, c_1, c_2);\qquad (13)$$

d)Normalization of set SNB' to the support determined under point a).

$$SL=\mathrm{supp}_L(B');\qquad SU=\mathrm{supp}_U(B');\qquad (14)$$

<u>The conclusion is</u>: $B'= \mathrm{suppnorm}_{SU}^{SL}(SNB')$; (15)

The conclusion is a regular fuzzy set in every case, since the normalization of the support must always result into a fuzzy set. Using this method we can get a conclusion for any type of fuzzy sets.

IV. SPECIALISED SOLID CUTTING METHOD

This method is a simplified version of the general method, which does not calculate all the points of the membership function, but only the four characteristic points. So this method is applicable for crisp, triangular and trapezoidal fuzzy sets, while it requires only a small amount of computational time. This method can also be divided into two parts. For simplicity let us denote A_3=A', A_4=A", B_3=B', B_4=B", and $a_{j,k}$ the k-th chararacteristic point of A_j in α, and $d_{Aj,k}$ the distance from $cp(A_j)$ (fig. 6).

1) Determination of A_4 and B_4:

$$d_{A,4,k} = d_{A,1,k} + (d_{A,2,k} - d_{A,1,k}) \cdot C \qquad (16)$$

$$d_{B,4,k} = d_{B,1,k} + (d_{B,2,k} - d_{B,1,k}) \cdot C \qquad (17)$$

where: k=1..4

$$C = \frac{cp(A_3) - cp(A_1)}{cp(A_2) - cp(A_1)} \qquad (18)$$

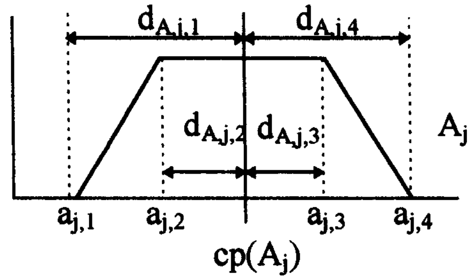


fig.6.

$$cp(B_3) = cp(B_4) = cp(B_1) + (cp(B_2) - cp(B_1)) \cdot C \qquad (19)$$

2) Determination of B_3 from A_3, A_4, B_4

a) Determination of Core(B_3)

$$b_{3,2}=F(a_{3,2};a_{4,2};b_{4,2};cp(A_3);cp(B_3)) \qquad (20)$$

$$b_{3,3} =cp(B_3)+F(d_{A,3,3};d_{A,4,3};d_{B,4,3};Mx-cp(A_3);My-cp(B_3)) \qquad (21)$$

b) Determination of the membership function of B_3 between $b_{3,1}$ and $b_{3,2}$: $\alpha \in [0,1]$;

$$\inf(B_{3\alpha}) = \inf(core(B_3)) \cdot F\left(\left[\inf(A_{3\alpha}) / \inf(core(A_3))\right];\left[\inf(A_{4\alpha}) / \inf(core(A_4))\right];\left[\inf(B_{4\alpha}) / \inf(core(B_4))\right];1;1\right) \qquad (22)$$

Set B' will be linear in this interval. In the same way linearity will hold between $b_{3,3}$ and $b_{3,4}$ also. (We omit the proof of this part due to the lack of space). Thus, it is enough to calculate the points $b_{3,1}$ and $b_{3,4}$ for α=0.

Determination of supp(B_3):

$$b_{3,1} = b_{3,2} \cdot F([\, a_{3,1} / a_{3,2} \,];\ [a_{4,1} / a_{4,2} \,];\ [b_{4,1} / b_{4,2} \,];\ 1;\ 1) \qquad (23)$$

Equation (23) is meaningless in an extreme case, when inf(core(A_3)) or inf(core(B_3))=0. Then, the problem can be eliminated by shifting the universe.

Determination of $b_{3,4}$ is done similary: where $Mx' \geq Mx$; $My' \geq My$;

$$b_{3,4}=My'-(My'-b_{3,3}) \cdot F([(Mx'-a_{3,4})/(Mx'-a_{3,3})];[(Mx'-a_{4,4})/(Mx'-a_{4,3})];[(My'-b_{4,4})/(My'-b_{4,3})];1;1); \qquad (23)$$

V. EXAMPLES

1) Examples for the general solid cutting method

The result of the general method can be seen in figure 7. One figure contains two diagrams. In the figures the observations are in the upper diagrams, and the conclusion fuzzy sets in the lower ones. The sets drawn with thin line are the fuzzy sets A" and B".

Figure a) shows a simple case, where the similarity of the conclusion fuzzy set to the observation set can be seen. Figure b) shows that the elements belonging mostly to sets B' and B" are the same, similarly to elements of sets A' and A". Figure c) shows a similar case to the previous one. Figure d) shows such a case, when the conclusion function can be hardly given using only human comprehension.

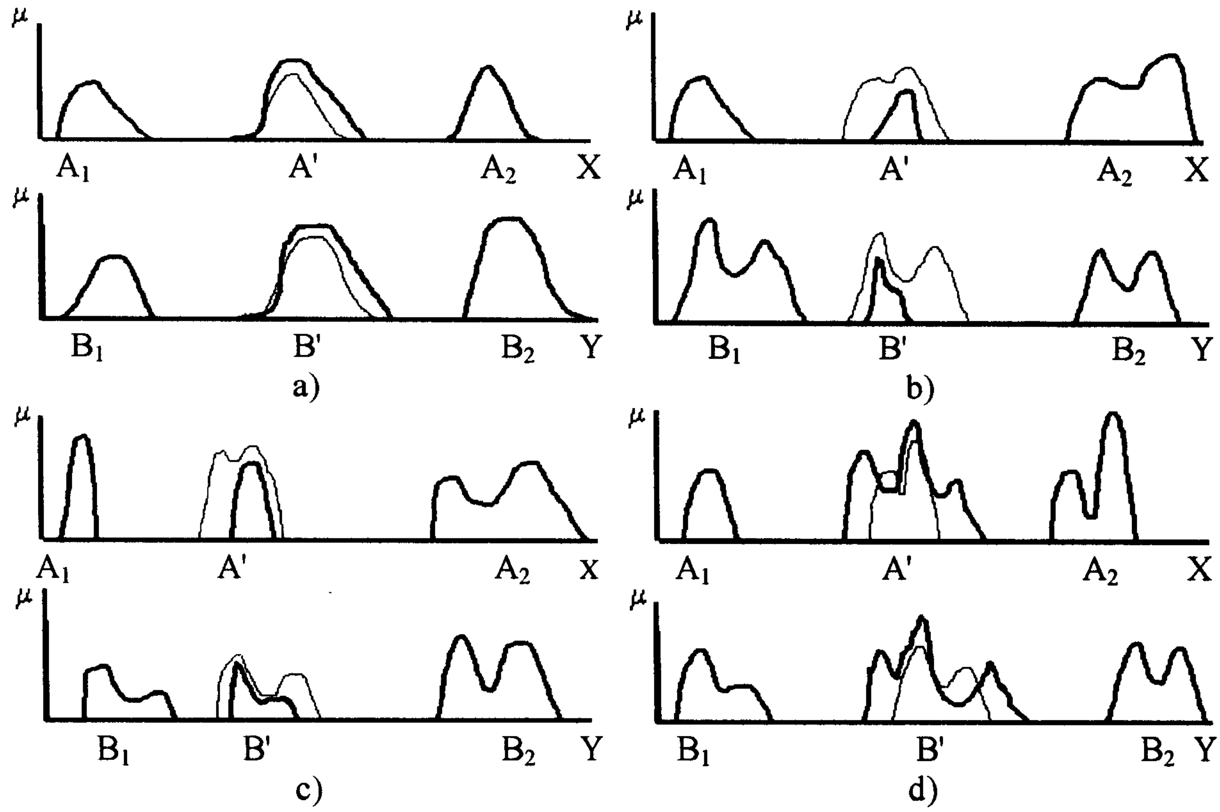


fig. 7

2) Examples for the specialised solid cutting method.

In figure 8 several examples A_1, A_2 and A' defined in X can be seen in the first column. The thin lined set is A". The next figures (columns 2 to 4) compare three different interpolation method. The second column treats the results obtained by the Kóczy Hirota interpolation (single linear method), and the third column contains results obtained by its extended version (Vass, Kalmár and Kóczy). The last column can be found the conclusion fuzzy sets B' generated by the specialized method.

In the first line the terms are rather "nice", and so every method results into a directly interpretable conclusion. It can be observed that, if all methods give an interpretable conclusion (there is no need for normalization) and only triangular fuzzy sets are used, the results are almost identical. The second line shows an example, with trapezoidal fuzzy sets, where the results in the second and third column are significantly different. The results in the third and fourth columns are more in accordance with the features of the observation. Comparing these three different methods it can be said that the Kóczy-Hirota-method usually gives an almost identical conclusion (if it is directly interpretable) with the newly introduced specialized method. The third and fourth lines present examples where the specialized method still gives directly interpretable conclusions while the others do not. (In the first case the "membership" function has to be regularized (by taking only its part that is alone the support, and the this truncated result has to be normalized, while in the second, the linear interpolation method results in to a loop that has to be cut, and again, normalization is needed). The use of a single distance measured from the centre points of the core guarantees the normality and the convexity.

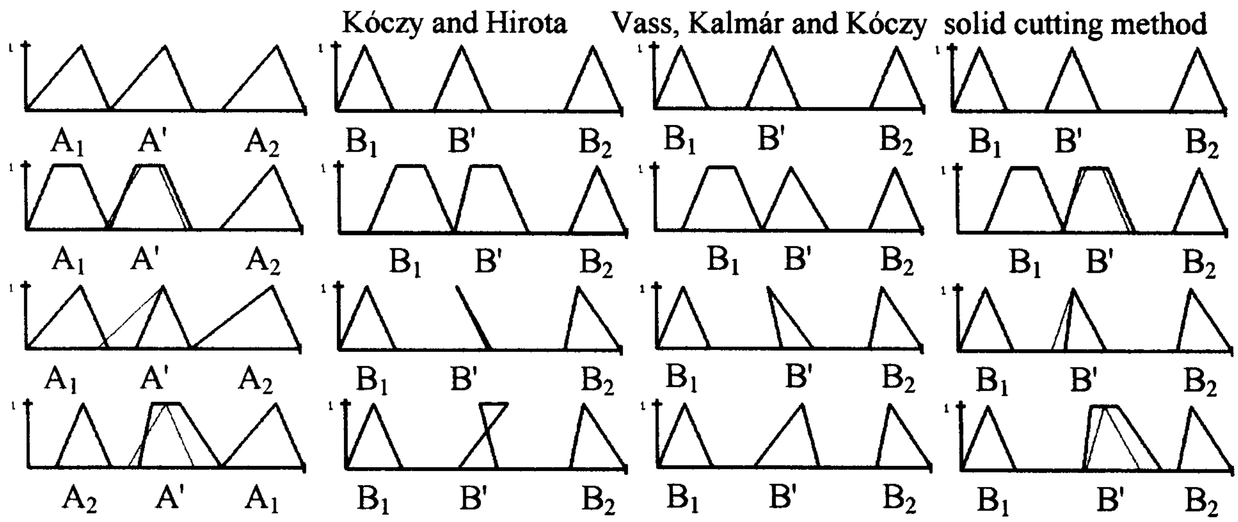


fig. 8

VI. CONCLUSION

The interpolation methods in the previous literature can be applied only on convex sets, while the generalized method introduced in this paper is applicable for arbitrary type fuzzy sets. As a matter of course, the general method makes calculation necessary for "every point" of the set. The specialized method can be applied on the most commonly used crisp, triangular and trapezoidal fuzzy sets, and requires only small computational time. The original linear and nonlinear interpolation methods present another problem, namely the conclusion set is not always convex, therefore it is not sufficient to calculate only the four characteristic points of the set. Our specialized method eliminates this problem, because it results always into crisp, triangular or trapezoidal conclusions, therefore it is enough to calculate only the four characteristic points. Another difference is that the new method offers a directly interpretable conclusion in every case. We are working on the multi-dimensional version of the method, where the conclusion can be obtained for observations on more than one component universes.

VII. REFERENCES

[1] L. A. Zadeh: Outline of a new approach to the analysis of complex system and decision processes. IEEE Tr. SMC (1973).
[2] E. H. Mamdani and S. Assilian: An experiment in linguistic synthesis with a fuzzy logic controller. Int. J. of Man-Machine Studies (1975).
[3] W. Pedrycz: Fuzzy controll and fuzzy systems. Second edition J. Wiley Chichster 1992
[4] L. T. Kóczy and K. Hirota Interpolation and size reduction in fuzzy rule bases. Technical Report TR 93-94/401 (1993).
[5] Gy. Vass, L. Kalmár and L.T.Kóczy: Extension of the fuzzy rule interpolation method. Int. Conf. on Fuzzy Sets Theory and Applications, Lipótszentmiklós (Czechoslovakia), 1992
[6] L.T.Kóczy and K. Hirota: Interpolative reasoning with insufficient evidence in sparse fuzzy rule bases. Information Sciences 71 (1993). 169-201.
[7] L. Ding, Z. Shen and M. Mukaidono: Principle for approximate reasoning - based on Revision Principle. 2nd Int. Conf. Fuzzy Logic and Neural Networks, IIZUKA '92, Izuka, 1992. 305-308.
[8] Yan Shi and Masaharu Mizumoto: Some Considerations on Kóczy's Fuzzy Interpolative Reasoning Method. The fourth IEEE International Conference on Fuzzy Systems and second International Fuzzy Engineering Symposyum, Yokohama, Japan, 1995.
[9] L.T.Kóczy and K.Hirota: Ordering, distance and doseness of fuzzy sets. Fuzzy Sets and Systems 59 (1993) North-Holland, pp 281-293.
[10] P.Baranyi, T.D.Gedeon and L.T.Kóczy: A general method for fuzzy rule interpolation: specialized for crisp triangular and trapezoidal rules. Third Europen Conference on Fuzzy and Intelligent Techniques, Aachen Germany, 1995. pp 99-103.