SHADOWED SETS:
REPRESENTING AND PROCESSING FUZZY SETS

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Abstract The study introduces a new concept of shadowed sets that can be regarded as a certain operational framework simplifying processing carried out with the aid of fuzzy sets and enhancing interpretation of results obtained therein. Some conceptual links between this idea and some others known in the literature are established. In particular, it is demonstrated how fuzzy sets can induce shadowed sets. Subsequently, shadowed sets point out at interesting algorithmic relationships existing between rough sets and fuzzy sets. Detailed computational aspects of shadowed sets are discussed. Several illustrative examples are provided as well.

Keywords fuzzy sets, shadowed sets, rough sets, symbolic computing, three - valued logic, vagueness, decision - making, fuzzy clustering

1. Introduction

Fuzzy sets are regarded as one of the formal vehicles devoted to capture, represent and processing vagueness [2] [4] [5] and thus allow to cope with diverse phenomena exhibiting unclearly defined boundaries. Do fuzzy sets live up to such expectations ? The answer is affirmative yet some outstanding questions remain. When it comes to representing vagueness, fuzzy sets tend to capture this exclusively via membership functions that are mappings from a given universe of discourse to a unit interval containing membership values. As always emphasized in the literature, a membership grade indicates an extent to which a given point in the universe of discourse belongs to a concept we are about to represent. Once the membership function has been established (estimated or defined), the concept is described very precisely as the membership values are exact numerical quantities. This seems to raise a certain dilemma which, in fact, has already sparked a lot of debates starting from the very inception of fuzzy sets.

The study is devoted to a certain different model of vagueness called shadowed sets which does not lend itself to precise numerical membership values but relies on basic concepts of truth values (yes - no) and an entire unit interval perceived as a zone of uncertainty.

Fuzzy sets are regarded as an emerging point of this discussion; it is revealed how shadowed sets can be induced by fuzzy sets by suppressing all detailed numerical information about membership values inherently associated with the latter. First we introduce a constructive way in which shadowed sets are developed based on fuzzy sets (Section 2). In Section 3 we discuss several examples highlighting the use of shadowed sets in decision making and image analysis.

2. From fuzzy sets to shadowed sets

Consider the problem of representing (or approximating) a fuzzy sets by a less precise and rough construct which does not require so much precision and calls for less computational effort. Several ways could be pursued; these have been reported in the existing literature. In particular:

a. Any fuzzy set can be approximated by a certain α - cut [4][5] meaning that all elements that belong to the fuzzy sets to a membership degree less than α are dropped whereas all these with the membership above this threshold are admitted to the resulting set. The appealing question is then the one about optimality of a specific level of the threshold (α).

b. One can restrict to a so-called level fuzzy set [8]. The intent of this construct is to preserve only the most significant membership values and eliminate those lying below the threshold level and therefore being perceived as completely meaningless. This modification reduces an amount of computing, and when tailored to problems of information retrieval allows to filter out a vast number of otherwise irrelevant records resulting as a response to a fuzzy query. Again, the same problem of selecting α remains open - the lower the value of α, the more original information present in A becomes retained. Notice that in both these cases a fuzzy set is replaced by another set or a part of the original fuzzy set resulting in a purely numeric structure.

It is interesting to underline that a purely numeric format of representing fuzzy sets has raised some concerns in the past and, in sequel, has triggered a search for some other models that are less numeric and precise. As already mentioned, this trend of questioning excessive precision of fuzzy sets has been motivated by the conceptual shortcoming associated with precise numeric values of membership used to describe vague concepts. The justification of this concern is also on an experimental side. Along this line let us recall several enhancements such as fuzzy sets of type -2 [6], interval valued fuzzy sets [10], and probabilistic sets [3], in particular. The underlying premise is that the grades of membership themselves are rather fuzzy sets, sets, or truncated random variables all being defined in the unit interval. In studies on probabilistic sets [3] (even though they originate on a somewhat different conceptual and computational ground), it has been found that most of uncertainty in the determination of the membership values is associated with those grades situated *around* 0.5. This finding is quite appealing. In contrast, we are usually far more confident about assigning values close 1 (thus counting the elements in) or 0 (therefore making the corresponding element excluded from the concept). On the other hand, the membership values (such as those *around* 0.5) always spark some hesitation and are always more difficult to place on a simple numeric scale. This observation forms a cornerstone of the model to be developed in this study. Consider a fuzzy set A. We elevate some membership values (usually those that are high enough) and reduce those that are viewed as substantially low. The elevation and reduction mechanisms are quite a radical as we go for 1 and 0, respectively. In other words, we can say that by doing that we eliminate (disambiguate) the original concept described by the fuzzy set. As this reduces vagueness we make some extra provisions for maintaining the overall level of vagueness constant by allowing some other regions of the universe of discourse associated with intermediate membership values that are defined in a more relaxed way or, simply, to let them be totally undefined. To do that, rather than attaching there a single specific membership value, we assign a unit interval that could be regarded as a nonnumeric model of membership grade. Fig. 1 summarizes the proposed construct - the shadowed set is induced from the fuzzy set by accepting a specific threshold level.

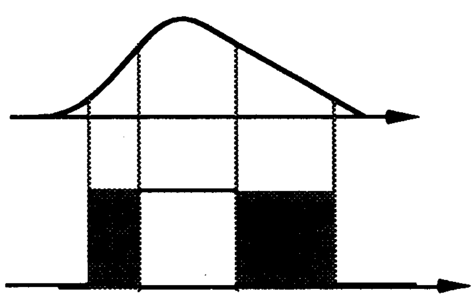

Fig. 1. Fuzzy sets and shadowed sets

Observe that this development produces an effect of vagueness allocation as the regions of vagueness (unit intervals) are allocated to some regions of **X** rather than to the entire space as encountered in fuzzy sets. In essence, we have transformed the fuzzy set to a set with some clearly marked vagueness zones or, put it more descriptively, shadows. Note that the result is a structure that maps **X** to 0, 1, and [0,1]. We will call this concept a shadowed set

$$\mathbb{A} : \mathbf{X} \rightarrow \{ 0, 1, [0,1] \} \quad (1)$$

The elements of **X** for which $\mathbb{A}$ attains 1 constitute its core while the elements where $\mathbb{A}(x) = [0,1]$ form a shadow of this construct. One can envision some particular cases such as a shadowed set without any core (only shadow available) and shadowed sets with nonexistent shadows.

To proceed with more computational issues, we address a balance of vagueness. As some of the regions come with elevated or reduced membership values (1 and 0), this process should happen at an expense of increased uncertainty in the intermediate membership values, namely an introduction of unit intervals to be distributed across some ranges of X. As shown in Fig. 2, we study the areas below the membership function and these need to be balanced by selecting a suitable threshold α meaning that the following relationship holds

$$V = \left| \int_{-\infty}^{a_1} A(x)dx + \int_{a_2}^{+\infty} (1-A(x))dx - \int_{a_1}^{a_2} A(x)dx \right| \quad (2)$$

i.e.,

$$\Omega_1 + \Omega_2 = \Omega_3$$

In other words, the threshold $\alpha \in [0, 1/2)$ should lead to $V(\alpha) = 0$.

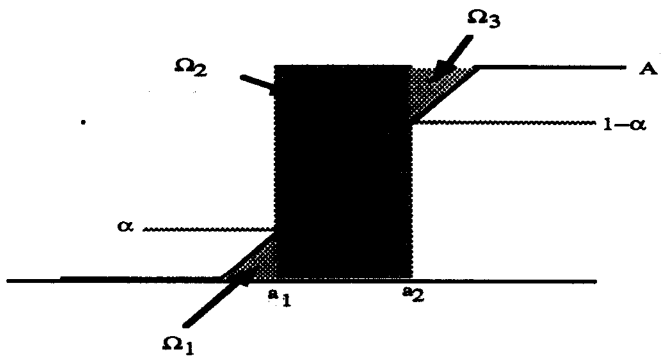

Fig. 2. Construction of a shadowed set

Shadowed sets exhibit some interesting conceptual links with the existing concepts, especially interval valued sets [10] and rough sets [7]. With respect to the first class, shadowed sets are somewhat subsumed by them. The important operational difference lies in the way in which these concepts have been formed. Interval - valued fuzzy sets are developed independently from fuzzy sets and, by no means are implied by them, whereas shadowed sets, as shown above, can be directly implied (induced) by fuzzy sets. Conceptually, shadowed sets are close to rough sets even though the mathematical foundations of these latter are very different. In rough sets we distinguish between three regions [7]:

- the regions whose elements are fully accepted (membership value equal 1) and belonging to the concept under discussion,
- the regions whose elements definitely do not belong to the concept

- the regions where membership grade is doubtful - these come in the form of the shadows of the introduced shadowed sets
In this sense shadowed sets narrow down a conceptual and an algorithmic gap between fuzzy sets and rough sets highlighting how these could be directly related.

Two points are worth underlining in the setting established so far:
-the proposed concept attempts to capture vagueness in a nonnumeric fashion - we do not commit ourselves to any specific (and precise) membership values over the specific regions of the universe of discourse.
-the factor of vagueness becomes localized in the form of shadows as opposed to the situation existing with fuzzy sets where it is spread across the entire universe of discourse.

For discrete universes of discourse when we are dealing with a collection of membership values rather than continuous functions, (2) involves several sums and emerges in the form

$$V = \left| \sum_{i:\, A(x_i) < \alpha} A(x_i) + \sum_{i:\, A(x_i) > 1-\alpha} (1 - A(x_i)) - \mathrm{card}\{ x_i \in \mathbf{X} \mid \alpha \le A(x) \le 1-\alpha \} \right| \tag{3}$$

Let us discuss computations of the threshold level α for some selected classes of continuous membership functions
1. Triangular membership functions. Assume the membership function of the form

$$y = \begin{cases} \frac{x-a}{b-a} & \text{for } x \in [a, b] \\ 0, & \text{otherwise} \end{cases}$$

For given α the values of a_1 and a_2 are equal

$$a_1 = a + \alpha(b-a)$$

and

$$a_2 = a + (1-\alpha)(b-a)$$

Computing the corresponding areas and solving the resulting quadratic equation with respect to α we derive

$$\alpha = \frac{2^{3/2} - 2}{2} = 0.414$$

and

$$\alpha = \frac{-2^{3/2} - 2}{2} = -2.414$$

Obviously, the only first root becomes accepted as it satisfies the requirement formulated in (3). Note that the level of α does not depend on the values of "a" and "b". These bounds appear in the formulas describing the shadow itself, namely

$$a_1 = a + 0.414(b-a) = 0.414b - 0.586a$$
$$a_2 = a + 0.586(b-a) = 0.586b - 0.414a$$

2. Consider the nonlinear membership function assuming the form

$$A(x) = \sqrt{\frac{x-a}{b-a}}$$

Here $a_1 = a + \alpha^2(b-a)$ and $a_2 = a + (1-\alpha)^2(b-a)$.
Proceeding in the same way as before, the problem gives produces a fifth order polynomial equation - the only root satisfying the imposed requirement is $\alpha = 0.405$.

3. Examples

3. 1. Image processing

The use of shadowed sets or shadowed relations can be also substantial in image processing especially in the problem of determining boundaries of objects. Consider a blurred circle as in Fig. 3. Due to a spectrum of different levels of brightness, this object can be conveniently regarded as a two argument fuzzy relation. The minimization of V leads to $\alpha = 0.395$, Fig. 4, and the produced residual structure (shadow of the fuzzy relation) identifying the boundary of the circle is visualized in Fig. 5.

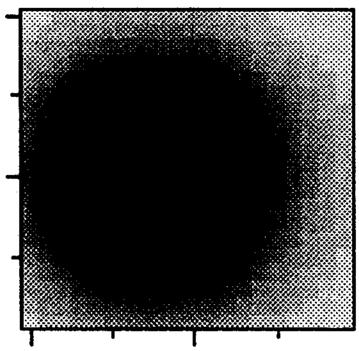

Fig. 3. Two dimensional image of a disk

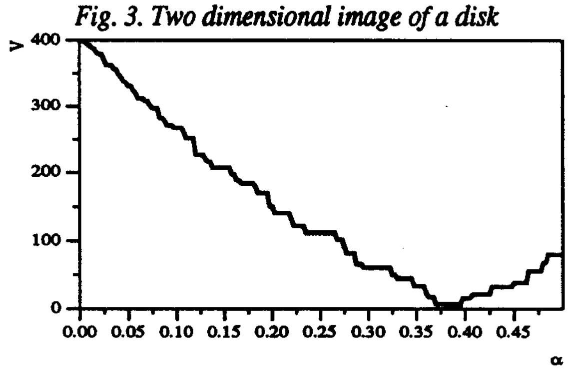

Fig. 4. V as a function of α

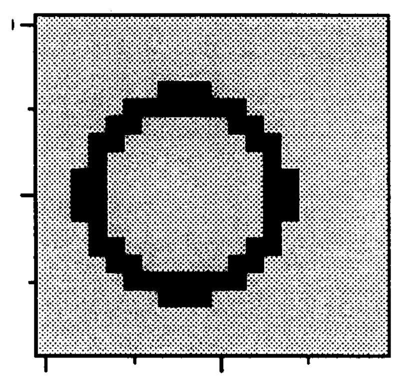

Fig. 5. Shadow of the shadowed set

3. 2. Single stage decision - making in presence of fuzzy objectives

The problem of a single stage decision - making has been a favorite example showing the use of fuzzy set technology in this area In the simplest scenario possible, the resulting fuzzy decision is regarded to be a fuzzy set formed as an intersection of several decision objectives (viewed as some goals and constraints) and treated as fuzzy sets defined in the same universe of discourse. For sake of conciseness, let us consider only two objectives (fuzzy sets), namely A and B defined in **X**. Rather than using these fuzzy sets directly and compute their intersection, we proceed with the corresponding shadowed sets, $\mathbb{A}$ and $\mathbb{B}$, and determine a shadowed set of decision meaning that

$$\mathbb{D} = \mathbb{A} \cap \mathbb{B}$$

Three interesting and qualitatively distinct cases arise:

a. The decision $\mathbb{D}$ is empty; $\mathbb{D}$ assumes zero over **X**. The analysis worked out with shadowed sets has evidently identified the case where no decision could be made - this is primarily because of strongly conflicting objectives in the decision problem. $\mathbb{A}$ and $\mathbb{B}$ are too distinct to advice making any rational decision. What it occurs when using fuzzy sets is that the resulting fuzzy set of decision D is highly subnormal which could still push us to make a decision by e.g., selecting the modal value of D.

b. In the second case $\mathbb{A}$ and $\mathbb{B}$ are getting closer each other and this results in the form of $\mathbb{D}$. The interpretation is also straightforward: "decide at your own risk" - there is no indication which subset of **X** could be legitimate as the shadowed set does not have any core. Perhaps the only choice worth considering will the region identified by the shadow of $\mathbb{D}$.

c. Finally, enough overlap of $\mathbb{A}$ and $\mathbb{B}$ (or less conflicting nature of these) has produced $\mathbb{D}$ - its core identifies potential decision values. Note, however, that further selection between the elements of the core of $\mathbb{D}$ is possible.

4. Conclusions

We have introduced and studied a new concept of shadowed sets. These constructs are viewed as being induced by fuzzy sets and aimed at less numeric and far more computationally demanding processing of information conveyed by fuzzy sets. This feature could be especially important in all situations where a certain trade-off between numeric precision and computational effort becomes necessary. Shadowed sets enhance and simplify an interpretation of results of processing with fuzzy sets by proposing decision expressed in the language of three - valued logic (that could be interpreted as yes, no, and unknown). As numeric details are suppressed while computing efficiency increased, one can think of shadowed sets as a provider of a quick and dirty approach to computing with fuzzy quantities - if the obtained results are of interest (usually shadowed sets with nonempty cores) then one can resort to detailed yet time consuming computing with fuzzy sets.

Acknowledgments

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5. References

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