GREY ALGEBRA AND GREY IDEAL

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Abstract: In this paper,the concept of a grey algebra over a grey field is introduced and the grey ideals over a grey field are discussed.
Keywords: Grey subset; Grey field; Grey algebra; Grey ideal.

1 Preliminaries

Defination 1.1 Let X be a nonempty set, a grey subset of X is a pair of maps $\underline{u}$ and $\bar{u}$ from X to closed interval $[0,1]$,where $\bar{u}\geqslant\underline{u}$,we write $[\underline{u},\bar{u}]$ for short.

Definition 1.2 Let $[\lambda_1,\lambda_2]$ and $[a_1,a_2]$ be two closed intervals contained in $[0,1]$,we define

(i) $[\lambda_1,\lambda_2]\geqslant[a_1,a_2] \iff \lambda_1\geqslant a_1, \lambda_2\geqslant a_2$

(ii) $[\lambda_1,\lambda_2]=[a_1,a_2] \iff \lambda_1=a_1, \lambda_2=a_2$

(iii) $[\lambda_1,\lambda_2]\wedge[a_1,a_2]=[\lambda_1\wedge a_1, \lambda_2\wedge a_2]$

(iv) $[\lambda_1,\lambda_2]\vee[a_1,a_2]=[\lambda_1\vee a_1, \lambda_2\vee a_2]$

Definition 1.3 Let $[\underline{u},\bar{u}]$ and $[\underline{v},\bar{v}]$ be two grey subsets of X, A a subset of X, we define

(i) $[\underline{u},\bar{u}](x) = [\underline{u}(x),\bar{u}(x)], \forall x\in X$

(ii) $\vee\{[\underline{u},\bar{u}](x): x\in A\} = [\vee\{[\underline{u}(x): x\in A\}, \vee\{\bar{u}(x): x\in A\}]$

(iii) $\{[\underline{u},\bar{u}]\wedge[\underline{v},\bar{v}]\}(x) = [\underline{u},\bar{u}](x)\wedge[\underline{v},\bar{v}](x)$

(iv) $\{[\underline{u},\bar{u}]\vee[\underline{v},\bar{v}]\}(x) = [\underline{u},\bar{u}](x)\vee[\underline{v},\bar{v}](x)$

2 Grey field and grey algebra

Definition 2.1 Let F be a field and $[\underline{u},\bar{u}]$ a grey subset of F. If the following conditions hold:

(i) $[\underline{u},\overline{u}](x+y) \geqslant [\underline{u},\overline{u}](x) \wedge [\underline{u},\overline{u}](y)$, $\forall x,y \in F$

(ii) $[\underline{u},\overline{u}](-x) \geqslant [\underline{u},\overline{u}](x)$, $\forall x \in F$

(iii) $[\underline{u},\overline{u}](xy) \geqslant [\underline{u},\overline{u}](x) \wedge [\underline{u},\overline{u}](y)$, $\forall x,y \in F$

(iv) $[\underline{u},\overline{u}](x^{-1}) \geqslant [\underline{u},\overline{u}](x)$, $\forall x \in F$, $x \neq 0$.

Then $[\underline{u},\overline{u}]$ is called a grey field of F.

It is clear that if $[\underline{u},\overline{u}]$ is a grey field of a field F then

(i) $[\underline{u},\overline{u}](-x) = [\underline{u},\overline{u}](x)$, $x \in F$

(ii) $[\underline{u},\overline{u}](x^{-1}) = [\underline{u},\overline{u}](x)$, $x \neq 0$.

Proposition 2.2 Let $[\underline{u},\overline{u}]$ be a grey subset of a field F. Then $[\underline{u},\overline{u}]$ is a grey field of F iff

(i) $[\underline{u},\overline{u}](x-y) \geqslant [\underline{u},\overline{u}](x) \wedge [\underline{u},\overline{u}](y)$, $\forall x,y \in F$

(ii) $[\underline{u},\overline{u}](xy^{-1}) \geqslant [\underline{u},\overline{u}](x) \wedge [\underline{u},\overline{u}](y)$, $\forall x,y \in F, y \neq 0$.

Proposition 2.3 Let $[\underline{u},\overline{u}]$ be a grey field of a field F, then

$[\underline{u},\overline{u}](0) \geqslant [\underline{u},\overline{u}](1) \geqslant [\underline{u},\overline{u}](x)$, $x \neq 0$, where 0 and 1 are zero element and identity of F respectively.

Definition 2.4 Let $[\underline{u},\overline{u}]$ be a grey field of a field F, X an algebra over F, $[\underline{v},\overline{v}]$ a grey subset of X. If for all $x,y \in X$, $\lambda \in F$

(i) $[\underline{v},\overline{v}](x+y) \geqslant [\underline{v},\overline{v}](x) \wedge [\underline{v},\overline{v}](y)$

(ii) $[\underline{v},\overline{v}](\lambda x) \geqslant [\underline{u},\overline{u}](\lambda) \wedge [\underline{v},\overline{v}](x)$

(iii) $[\underline{v},\overline{v}](xy) \geqslant [\underline{v},\overline{v}](x) \wedge [\underline{v},\overline{v}](y)$

(iv) $[\underline{u},\overline{u}](1) \geqslant [\underline{v},\overline{v}](x)$

Then $[\underline{v},\overline{v}]$ is called a grey algebra over the grey field $[\underline{u},\overline{u}]$.

Proposition 2.5 Let $[\underline{v},\overline{v}]$ be a grey field of a field F, X an algebra over F and $[\underline{v},\overline{v}]$ a grey subset of X. Then $[\underline{v},\overline{v}]$ is a grey algebra over the grey field $[\underline{u},\overline{u}]$ iff for all $x_1,x_2 \in X$, $\lambda_1, \lambda_2 \in F$

(i) $[\underline{v},\overline{v}](\lambda_1 x_1+\lambda_2 x_2) \geqslant \{[\underline{u},\overline{u}](\lambda_1) \wedge [\underline{v},\overline{v}](x_1)]\} \wedge \{[\underline{u},\overline{u}](\lambda_2) \wedge [\underline{v},\overline{v}](x_2)]\}$

(ii) $[\underline{v},\overline{v}](x_1 x_2) \geqslant [\underline{v},\overline{v}](x_1) \wedge [\underline{v},\overline{v}](x_2)]$

(iii) $[\underline{u},\overline{u}](1) \geqslant [\underline{v},\overline{v}](x_1)$

Definition 2.6 Let $[\underline{u},\overline{u}]$ be a grey field of a field F, X an algebra over F and $[\underline{v},\overline{v}]$ a grey algebra of X over the grey field $[\underline{u},\overline{u}]$. If for all $x, y \in X$

$[\underline{v},\overline{v}](xy) \geqslant [\underline{v},\overline{v}](x) \vee [\underline{v},\overline{v}](y)$

then $[\underline{v},\overline{v}]$ is called a grey ideal of X over the grey field $[\underline{u},\overline{u}]$.

Proposition 2.7 Let X and Y be two algebras over a field F, $[\underline{u},\overline{u}]$ a grey field of F, f an algebraic homomorphism of X into Y. If $[\underline{v},\overline{v}]$ is a grey algebra (grey ideal) of Y over the grey field $[\underline{u},\overline{u}]$, then the inverse $f^{-1}[\underline{v},\overline{v}]$ is a grey algebra (grey ideal) of X over the grey field $[\underline{u},\overline{u}]$, where

$f^{-1}[\underline{v},\overline{v}](x) = [\underline{v},\overline{v}](f(x)), \ \forall x \in X.$

Proof. We only prove the case of a grey algebra.

$\forall x_1, x_2 \in X, \ \forall \lambda_1, \lambda_2 \in F$

$f^{-1}[\underline{v},\overline{v}](\lambda_1 x_1 + \lambda_2 x_2) = [\underline{v},\overline{v}](f(\lambda_1 x_1 + \lambda_2 x_2))$

$= [\underline{v},\overline{v}](\lambda_1 f(x_1) + \lambda_2 f(x_2))$

$\geqslant \{[\underline{u},\overline{u}](\lambda_1) \wedge [\underline{v},\overline{v}](f(x_1))\} \wedge \{[\underline{u},\overline{u}](\lambda_2) \wedge [\underline{v},\overline{v}](f(x_2))\}$

$= \{[\underline{u},\overline{u}](\lambda_1) \wedge f^{-1}[\underline{v},\overline{v}](x_1)\} \wedge \{[\underline{u},\overline{u}](\lambda_2) \wedge f^{-1}[\underline{v},\overline{v}](x_2)\}$

$f^{-1}[\underline{v},\overline{v}](x_1 x_2) = [\underline{v},\overline{v}](f(x_1 x_2)) = [\underline{v},\overline{v}](f(x_1) f(x_2))$

$\geqslant [\underline{v},\overline{v}](f(x_1)) \wedge [\underline{v},\overline{v}](f(x_2))$

$= f^{-1}[\underline{v},\overline{v}](x_1) \wedge f^{-1}[\underline{v},\overline{v}](x_2)$

$[\underline{u},\overline{u}](1) \geqslant [\underline{v},\overline{v}](f(x_1)) = f^{-1}[\underline{v},\overline{v}](x_1)$

So, $f^{-1}[\underline{v},\overline{v}]$ is a grey algebra over the grey field $[\underline{u},\overline{u}]$.

Proposition 2.8 Let X and Y be two algebras over a field F, f an algebraic homomorphism of X into Y. If $[\underline{v},\overline{v}]$ is a grey algebra (grey ideal) of X over the grey field $[\underline{u},\overline{u}]$, then the image $f[\underline{v},\overline{v}]$ is a grey algebra (grey ideal) of Y over the grey field $[\underline{u},\overline{u}]$, where

$$f[\underline{v},\overline{v}](y) = \begin{cases} \vee\{[\underline{v},\overline{v}](x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \phi \\ [0,0] & \text{otherwise} \end{cases}$$

Proof. We only prove the case of a grey algebra.

$\forall y_1, y_2 \in Y$, If $f^{-1}(y_1 - y_2) = \phi$, then

$f[\underline{v},\overline{v}](y_1 - y_2) = [0,0] = f[\underline{v},\overline{v}](y_1) \wedge f[\underline{v},\overline{v}](y_2)$

Otherwise $f[\underline{v},\overline{v}](y_1 - y_2) = \vee\{[\underline{v},\overline{v}](x_1 - x_2) : x_1 - x_2 \in f^{-1}(y_1 - y_2)\}$

$\geqslant \vee \{[\underline{v},\overline{v}](x_1) \wedge [\underline{v},\overline{v}](x_2) : x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}$

$= \vee \{[\underline{v},\overline{v}](x_1) : x_1 \in f^{-1}(y_1)\} \wedge \vee \{[\underline{v},\overline{v}](x_2) : x_2 \in f^{-1}(y_2)\}$

$= f[\underline{v},\overline{v}](y_1) \wedge f[\underline{v},\overline{v}](y_2)$

In like manner, $f[\underline{v},\overline{v}](y_1 y_2) \geqslant f[\underline{v},\overline{v}](y_1) \wedge f[\underline{v},\overline{v}](y_2)$

If $f^{-1}(y_1) = \phi$, then $f[\underline{v},\overline{v}](y_1) = [0,0]$

Otherwise for all $x \in f^{-1}(y_1)$, we have $[\underline{u},\overline{u}](1) \geqslant [\underline{v},\overline{v}](x)$

Hence, $[\underline{u},\overline{u}](1) \geqslant \vee \{[\underline{v},\overline{v}](x) : x \in f^{-1}(y_1)\} = f[\underline{v},\overline{v}](y_1)$

So, $f[\underline{v},\overline{v}]$ is a grey algebra of Y over the grey field $[\underline{u},\overline{u}]$.

3 Operations on grey ideals

Proposition 3.1 Let $[\underline{u},\overline{u}]$ be a grey field of a field F, X an algebra over F, $[\underline{v}_1,\overline{v}_1]$ and $[\underline{v}_2,\overline{v}_2]$ grey algebras (grey ideals) of X over the grey field $[\underline{u},\overline{u}]$. Then $[\underline{v}_1,\overline{v}_1] \wedge [\underline{v}_2,\overline{v}_2]$ is a grey algebra (grey ideal) of X over the grey field $[\underline{u},\overline{u}]$.

Proof. We only prove the case of a grey algebra.

Let $[\underline{v},\overline{v}] = [\underline{v}_1,\overline{v}_1] \wedge [\underline{v}_2,\overline{v}_2]$

$\forall x_1, x_2 \in X$, $\forall \lambda_1, \lambda_2 \in F$

$[\underline{v},\overline{v}](\lambda_1 x_1 + \lambda_2 x_2) = [\underline{v}_1,\overline{v}_1](\lambda_1 x_1 + \lambda_2 x_2) \wedge [\underline{v}_2,\overline{v}_2](\lambda_1 x_1 + \lambda_2 x_2)$

$\geqslant \{[\underline{u},\overline{u}](\lambda_1) \wedge [\underline{v}_1,\overline{v}_1](x_1) \wedge [\underline{u},\overline{u}](\lambda_2) \wedge [\underline{v}_1,\overline{v}_1](x_2)\}$

$\wedge \{[\underline{u},\overline{u}](\lambda_1) \wedge [\underline{v}_2,\overline{v}_2](x_1) \wedge [\underline{u},\overline{u}](\lambda_2) \wedge [\underline{v}_2,\overline{v}_2](x_2)\}$

$= \{[\underline{u},\overline{u}](\lambda_1) \wedge [\underline{v},\overline{v}](x_1)\} \wedge \{[\underline{u},\overline{u}](\lambda_2) \wedge [\underline{v},\overline{v}](x_2)\}$

Similarly, $[\underline{v},\overline{v}](x_1 x_2) \geqslant [\underline{v},\overline{v}](x_1) \wedge [\underline{v},\overline{v}](x_2)$

And $[\underline{u},\overline{u}](1) \geqslant [\underline{v},\overline{v}](x)$ is obvious.

So, $[\underline{v},\overline{v}]$ is a grey algebra over the grey field $[\underline{u},\overline{u}]$.

Proposition 3.2 Let $[\underline{u},\overline{u}]$ be a grey field of a feild F and X an algebra over F. If $[\underline{v}_1,\overline{v}_1]$ and $[\underline{v}_2,\overline{v}_2]$ are two grey ideals of X over the grey field $[\underline{u},\overline{u}]$, then $[\underline{v}_1,\overline{v}_1] \oplus [\underline{v}_2,\overline{v}_2]$ is a grey ideal of X over the grey field $[\underline{u},\overline{u}]$, where

$\{[\underline{v}_1,\overline{v}_1] \oplus [\underline{v}_2,\overline{v}_2]\}(x) = \vee \{[\underline{v}_1,\overline{v}_1](x_1) \wedge [\underline{v}_2,\overline{v}_2](x_2) : x_1 + x_2 = x\}$

Proof. Let $[\underline{v},\overline{v}] = [\underline{v}_1,\overline{v}_1] \oplus [\underline{v}_2,\overline{v}_2]$

$\forall x,y \in X, \forall \lambda \in F$

$[\underline{v},\overline{v}](x+y) \geqslant \vee\{[\underline{v}_1,\overline{v}_1](x_1+x_2)\wedge[\underline{v}_2,\overline{v}_2](y_1+y_2): x_1+y_1=x, x_2+y_2=y\}$

$\geqslant \vee\{[\underline{v}_1,\overline{v}_1](x_1)\wedge[\underline{v}_2,\overline{v}_2](y_1): x_1+y_1=x\}$

$\wedge\vee\{[\underline{v}_1,\overline{v}_1](x_2)\wedge[\underline{v}_2,\overline{v}_2](y_2): x_2+y_2=y\}$

$= [\underline{v},\overline{v}](x)\wedge[\underline{v},\overline{v}](y)$

$[\underline{v},\overline{v}](xy) \geqslant \vee\{[\underline{v}_1,\overline{v}_1](x_1y)\wedge[\underline{v}_2,\overline{v}_2](x_2y): x_1+x_2=x\}$

$\geqslant \vee\{[\underline{v},\overline{v}](x_1)\wedge[\underline{v}_2,\overline{v}_2](x_2): x_1+x_2=x\}$

$= [\underline{v},\overline{v}](x)$

Similarly, $[\underline{v},\overline{v}](xy) \geqslant [\underline{v},\overline{v}](y)$

That is $[\underline{v},\overline{v}](xy) \geqslant [\underline{v},\overline{v}](x)\vee[\underline{v},\overline{v}](y)$

$[\underline{v},\overline{v}](\lambda x) \geqslant \vee\{[\underline{v}_1,\overline{v}_1](\lambda x_1)\wedge[\underline{v}_2,\overline{v}_2](\lambda x_2): x_1+x_2=x\}$

$\geqslant \vee\{[\underline{u},\overline{u}](\lambda)\wedge[\underline{v}_1,\overline{v}_1](x_1)\wedge[\underline{v}_2,\overline{v}_2](x_2): x_1+x_2=x\}$

$= [\underline{u},\overline{u}](\lambda)\wedge\vee\{[\underline{v}_1,\overline{v}_1](x_1)\wedge[\underline{v}_2,\overline{v}_2](x_2): x_1+x_2=x\}$

$= [\underline{u},\overline{u}](\lambda)\wedge[\underline{v},\overline{v}](x)$

$[\underline{u},\overline{u}](1) \geqslant [\underline{v},\overline{v}](x)$ is obvious.

So, $[\underline{v}_1,\overline{v}_1]\oplus[\underline{v}_2,\overline{v}_2]$ is a grey ideal of X over the grey field $[\underline{u},\overline{u}]$.

Proposition 3.3 Let $[\underline{u},\overline{u}]$ be a grey field of a field F and X an algebra over F. If $[\underline{v}_1,\overline{v}_1]$, $[\underline{v}_2,\overline{v}_2]$ and $[\underline{v}_3,\overline{v}_3]$ are grey ideals of X over the grey field $[\underline{u},\overline{u}]$, then

(i) $[\underline{v}_1,\overline{v}_1]\oplus[\underline{v}_2,\overline{v}_2] = [\underline{v}_2,\overline{v}_2]\oplus[\underline{v}_1,\overline{v}_1]$

(ii) $\{[\underline{v}_1,\overline{v}_1]\oplus[\underline{v}_2,\overline{v}_2]\}\oplus[\underline{v}_3,\overline{v}_3]=[\underline{v}_1,\overline{v}_1]\oplus\{[\underline{v}_2,\overline{v}_2]\oplus[\underline{v}_3,\overline{v}_3]\}$

This proposition is strightforward.

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