Interval—valued fuzzy subgroups induced by T—triangular norms

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Abstract: In this paper, we define interval—valued fuzzy subgroups and normal interval—valued fuzzy subgroups by T—triangular norms. At the same time, their properties are studied respectively. Furthermore, we obtain some results similar to the theories of classical groups.

Keywords: Interval—valued fuzzy sets; T—norms; interval—vallued fuzzy subgroups; normal interval—valued fuzzy subgroups.

1 Introduction

Interval—valued fuzzy sets were introduced by many authors [3—4] in 1975. In recent years, investigation for interval—valued fuzzy sets is rising with days [5, 6, 7, 8]. Because the membership functions of a fuzzy set isn't usually easy to be determined in the applications, but the membership degree of an interval—valued fuzzy set is relative easy to be determined. Meng [7] and Sun [8] studied their basis theories in detail, and obtain three basis theorems of interval—valued fuzzy sets, i. e., the decomposition theorem, the representation theorem and the extension principle.

Rosenfeld's [2] first presented the concept of fuzzy subgroups in 1971. R. Biswas [1] introduced and studied interval—valued fuzzy subgroups at the first time in 1994. Nevertheless, the two kinds of fuzzy subgroups were defined by minimal operator $\wedge$.

The main goal of this paper is to define the interval—valued fuzzy subgroups and the normal interval—valued fuzzy subgroups once again by T—triangular norms. And give a necessary and sufficient condition for interval—valued fuzzy sets to be the two kinds of fuzzy subgroups respectively. Finally, we will get the theorem of homomorphic images (inverse—images) by the extension principle of the interval—valued fuzzy sets.

In Section 2, we state the operations of the interval numbers and the definition of the interval—valued fuzzy sets. We also recall the definition of the T—triangular norms and some results about them. In Section 3 and 4, the interval—valued fuzzy subgroup and the normal interval—valued fuzzy subgroup is defined respectively. In addition, their properties and structures are discussed. In fact, this paper is a continuation and development of [1].

2 preliminaries and propositions

In this Section, we first give below some preliminaries.

Throughout this paper, let I be a closed unit interval, i. e., $I = [0,1]$.

Let $[I] = \{ \bar{a} = [a^-, a^+] ; a^- \leqslant a^+, a^-, a^+ \in I\}$. Especially, for arbitrary $a \in I$. putting $a = [a, a]$, then $a \in [I]$ is obvious. The elements in $[I]$ is called the interval numbers on I. we make the following definitions about interval numbers.

For any $\bar{a}_j \in [I], \bar{a}_j = [a_j^-, a_j^+], a_j^-, a_j^+ \in I, j \in J.$ we define

$\bigvee_{j\in J} a_j^- = \sup\{a_j^- ; j \in J\}$; $\bigwedge_{j\in J} a_j^- = \inf\{a_j^- ; j \in J\}$;

$\sup \bar{a}_j = \bigvee_{j\in J} [a_j^-, a_j^+] = [\bigvee_{j\in J} a_j^-, \bigvee_{j\in J} a_j^+]$; $\inf \bar{a}_j^- = \bigwedge_{j\in J} [a_j^-, a_j^+] = [\bigwedge_{j\in J} a_j^-, \bigwedge_{j\in J} a_j^+]$.

Especially, for $\bar{a}, \bar{b} \in [I]$, where $\bar{a} = [a^-, a^+]$, $\bar{b} = [b^-, b^+]$, we define

$\bar{a} = \bar{b}$ iff $a^- = b^-, a^+ = b^+$; $\bar{a} \leqslant \bar{b}$ iff $a^- \leqslant b^-, a^+ \leqslant b^+$; $\bar{a} < \bar{b}$ iff $\bar{a} \leqslant \bar{b}$ and $\bar{a} \neq \bar{b}$;

Clearly, $([I], \leqslant, \vee, \wedge)$ constitutes a complete lattice with a minimal element $\bar{0} = [0,0]$ and a maximal element $\bar{1} = [1,1]$.

Definition 2.1 Let X be an ordinary set, mapping $\bar{A}; X \to [I]$ is called an interval—valued fuzzy set (in short written by IVFS) on X. Let $IF(X)$ denote the family of all IVFS on X. For each $\bar{A} \in IF(X)$, suppose $\bar{A}(x) = [A^-(x), A^+(x)]$ where $A^-(x) \leqslant A^+(x)$, $x \in X$. Then the ordinary fuzzy set $\bar{A}; X \to I$ *and* $A^+; X \to I$ is called a lower fuzzy set and a upper fuzzy set of $\bar{A}$ respectively. In addition, we define.

$\Phi(x) = [0,0]$, $X(x) = [1,1]$ where $x \in X$. obviously, $\Phi, X \in IF(X)$.

for every $[\lambda_1, \lambda_2] \in [I]$, let $\bar{A}_{[\lambda_1,\lambda_2]} = \{x \in X; A^-(x) \geqslant \lambda_1, A^+(x) \geqslant \lambda_2\}$.

Then $\bar{A}_{[\lambda_1,\lambda_2]}$ is called a cut—set of $\bar{A}$. Evidently $\bar{A}_{[\lambda_1,\lambda_2]} = A^-_{\lambda_1} \cap A^+_{\lambda_2}$.

Definition 2.2 Mapping T; $[0,1]^2 \to [0,1]$ is called a T—triangular norm (in short written by T—norm), if the following conditions are satisfied. where $a,b,c \in [0,1]$.

(1) $T(a,1) = a$

(2) $T(a,b) = T(b,a)$

(3) $T(a,b) \leqslant T(a,c)$ *if* $b \leqslant c$.

(4) $T(a,T(b,c)) = T(T(a,b),c)$.

Especially, Let T_H be a T—norm with $T(a,a) = a$, where $a \in [0,1]$. Then we call T_H an idempotent T—norm.

Definition 2.3 Let T be a T—norm, $\bar{a}$, $\bar{b} \in [I]$, $\bar{a} = [a^-, a^+]$, $\bar{b} = [b^-, b^+]$.

Let T; $[I] \times [I] \to [I]$. we define $T(\bar{a},\bar{b}) = [T(a^-,b^-), T(a^+,b^+)]$.

Proposition 2.1 Let T be a T—norm, then for arbitrary $a,b \in [0,1]$, we have

(1) $T(a,0) = 0$

(2) $T(a,b) \leqslant min\{a,b\}$.

Definition 2.4 Let T_1 *and* T_2 are T—norms. Then T_1 is called to be stronger than T_2, if for each a, $b \in [0,1]$, $T_1(a,b) \geqslant T_2(a,b)$ holds. written as $T_1 \geqslant T_2$.

Proposition 2.2. Let T be a T—norm. Then for every $a \in [0,1]$, $T \leqslant \wedge$ *and* $T = \wedge$ iff $T(a,a) = a$.

3 Interval—valued fuzzy subgroups

In this section, we first give the extension principle in IVFS. Next, we define the interval—valued fuzzy subgroup once again by operator T—norms. consequently·we will obtain the main results in this section, i. e., a necessary and sufficient condition for IVFS on groups to be interval—valued fuzzy subgroups and homomorphic theorem with respect to them.

Definition 3.1 Let X and Y be two ordinary sets. mapping $f: X \to Y$ induces two mappings $F_f: IF(X) \to IF(Y)$ and $F_f^{-1}: IF(Y) \to IF(X)$. We define

$$F_f(\overline{A})(y) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} \overline{A}(x) & \text{if } f^{-1}(y) \neq \Phi, y \in Y. \\ [0,0] & \text{otherwise.} \end{cases} \quad \text{and} \quad F_f^{-1}(\overline{B})(x) = \overline{B}(f(x))$$

where $f^{-1}(y) = \{x \in X; f(x) = y\}$, $\overline{A}, \overline{B} \in IF(X)$.
This is a process which the mapping between IVFS is transformed by an ordinary mapping f. We call the process the extension principle of IVFS. The mapping F_f and F_f^{-1} is called an interval—valued fuzzy transformation and inverse transformation induced by f respectively.

Clearly, we have

$$F_f(\overline{A})(y) = [\bigvee_{x\in f^{-1}(y)} A^-(x), \bigvee_{x\in f^{-1}(y)} A^+(x)] = [F_f(A^-)(y), F_f(A^+)(y)]$$

$$F_f^{-1}(\overline{B})(x) = [B^-(f(x)), B^+(f(x))] = [F_f^{-1}(B^-)(x), F_f^{-1}(B^+)(x)]$$

They are simply written as $F_f(\overline{A}) = [F_f(A^-), F_f(A^+)]$ *and*

$$F_f^{-1}(\overline{B}) = [F_f^{-1}(B^-), F_f^{-1}(B^+)].$$

Definition 3.2 Let G be a classical group, T be a T—norm. Then the IVFS $\overline{A}; G \to [I]$ is called an interval—valued fuzzy subgroup (in short written by IVFG')with respect to T, if the following conditions are fulfilled

(1) $\overline{A}(x \cdot y) \geqslant T(\overline{A}(x), \overline{A}(y))$

(2) $\overline{A}(x^{-1}) \geqslant \overline{A}(x)$

where $x, y \in G$.

Let $IF(G', T)$ denote the family of all IVFG' on G.

Note: If $T = \wedge$, IVFG' in this paper is exactly IVFG' in [1].

Theorem 3.1. Let G be a classical group, T be a T—norm, $\overline{A}$ be an IVFS on G and $\overline{A}(e) = [1,1]$ (e is a unit element of G). Then

$\overline{A} \in IF(G', T)$ iff for all $x, y \in G$, $\overline{A}(x \cdot y^{-1}) \geqslant T(\overline{A}(x), \overline{A}(y))$.

proof Necessity. It is clearly veryfied by the monotonicity of T—norms.
Sufficiency. First, for arbitrary $x \in G$, we have

$$\begin{aligned} \overline{A}(x^{-1}) = \overline{A}(e \cdot x^{-1}) \geqslant T(\overline{A}(e), \overline{A}(x)) &= [T(1, A^-(x)), T(1, A^+(x))] \\ &= [A^-(x), A^+(x)] \\ &= A(x) \end{aligned}$$

Second, for every $x, y \in G$, we can infer that

$$\overline{A}(x \cdot y) = \overline{A}(x \cdot (y^{-1})^{-1}) \geqslant T(\overline{A}(x), \overline{A}(y^{-1})) \geqslant T(\overline{A}(x), \overline{A}(y))$$

Consequently, $\overline{A} \in IF(G', T)$

Theorem 3.2 Let G be a classical group, T_1, T_2 be T—norms and $T_1 \geqslant T_2$.
Then $IF(G', T_1) \subset IF(G', T_2)$

proof obvious.

Corollary Let G be a classical group, T be a T —norm. Then

(1)$IF(G^{*}, \wedge) \subset IF(G^{*},T)$

(2)$IF(G^{*}, \cdot) \subset IF(G^{*}, \vee)$. where $\cdot$is a multiplicative operator.

Note: Clearly, above IVFG* is a generalization of the IVFG* in [1]

Theorem 3.3 Let G and $\bar{G}$ be classical groups, mapping $f: G \to \bar{G}$ be a homomorphism of the groups, T be a T —norm. Then the following conclusions hold.

(1) *If* $\bar{B} \in IF(\bar{G}^{*},T)$, *then* $F_f^{-1}(\bar{B}) \in IF(G^{*},T)$.

(2) If T is a continous T —norm and $\bar{A} \in IF(G^{*},T)$. Then $F_f(\bar{A}) \in IF(\bar{G}^{*},T)$.

proof We only proof (1), on the one hand, for each $x \in G$, by definition 3.1, 3.2 and homomorphic definition, we can infer that

$F_f^{-1}(\bar{B})(x^{-1}) = \bar{B}(f(x^{-1})) = \bar{B}((f(x))^{-1}) \geqslant \bar{B}(f(x)), = F_f^{-1}(\bar{B})(x)$

On the other hand, for every $x,y \in G$, we can get that

$F_f^{-1}(\bar{B})(x \cdot y) = \bar{B}(f(x \cdot y)) = \bar{B}(f(x) \cdot f(y)) \geqslant T(\bar{B}(f(x)), \bar{B}(f(y)))$

$= T(F_f^{-1}(\bar{B})(x), F_f^{-1}(\bar{B})(y))$.

Therefore $F_f^{-1}(\bar{B}) \in IF(G^{*},T)$.

4 Normal interval-valued fuzzy subgroups

In this section, we define the normal interval—valued fuzzy subgroups on groups, obtain a necessary and sufficient condition for IVFS to be normal interval—valued fuzzy subgroups. Furthermore, we get the theorem of homomorphic images (inverse—images).

Definition 4.1 Let G be a classical group, T be a T —norm. $\bar{A} \in IF(G^{*},T)$. $\bar{A}(x \cdot y) = \bar{A}(y \cdot x)$ for all $x,y \in G$. Then $\bar{A}$ is called a normal interval—valued fuzzy subgroup (in short written by NIFG*) with respect to T on G.

Let $NIF(G^{*},T)$ denote the family of NIFG* with respect to T —norms on G.

Theorem 4.1 Let G be a classical group, T be a T —norm, $\bar{A} \in IF(G^{*},T)$ *and* $\bar{A}_{[\lambda_1,\lambda_2]}$ be a subgroup in G for all $[\lambda_1,\lambda_2] \in [I]$. Then $\bar{A} \in NIF(G^{*},T)$ iff $\bar{A}_{[\lambda_1,\lambda_2]}$ is a normal subgroup in G.

proof Necessity. For any $x \in A_{[\lambda_1,\lambda_2]} = A_{\lambda_1}^{-} \cap A_{\lambda_2}^{+}$, $y \in G, [\lambda_1,\lambda_2] \in [I]$.

We have $\bar{A}(y \cdot x \cdot y^{-1}) = \bar{A}(y^{-1} \cdot (y \cdot x)) = \bar{A}(x) = [A^{-}(x), A^{+}(x)] \geqslant [\lambda_1,\lambda_2]$.

It follows that $y \cdot x \cdot y^{-1} \in \bar{A}_{[\lambda_1,\lambda_2]}$. i. e., $\bar{A}_{[\lambda_1,\lambda_2]}$ is a normal subgroup in G.

Sufficiency. Suppose $\bar{A} \notin NIF(G^{*},T)$, it means that there exists x_0, $y_0 \in G$, such that $\bar{A}(x_0, y_0) \neq \bar{A}(y_0 \cdot x_0)$. Without loss of generality, we assume that

$\bar{A}(x_0 \cdot y_0) < \bar{A}(y_0 \cdot x_0)$, and let $[\lambda_1,\lambda_2] = \frac{1}{2}(\bar{A}(x_0 \cdot y_0) + \bar{A}(y_0 \cdot x_0))$.

Evidently, we have $\bar{A}(x_0 \cdot y_0) < [\lambda_1,\lambda_2] \leqslant \bar{A}(y_0 \cdot x_0)$.

It follows that $x_0 \cdot y_0 \notin \bar{A}_{[\lambda_1,\lambda_2]}$ and $y_0 \cdot x_0 \in \bar{A}_{[\lambda_1,\lambda_2]}$.

Thus $y_0^{-1} \cdot (y_0 \cdot x_0) \cdot y_0 = x_0 \cdot y_0 \notin \overline{A}_{(\lambda_1,\lambda_2)}$.
This contradicts that $\overline{A}_{(\lambda_1,\lambda_2)}$ is a normal subgroup in G.

Corollary 4.1 Let G be a classical group, T be a T —norm, $\overline{A} \in NIF(G^{\cdot},T)$.
Let $G|_{\overline{A}} = \{ x \in G: \overline{A}(x) = \overline{A}(e)\}$,where e is a unit element of G.
Then the classical set $G|_{\overline{A}}$ is a normal subgroup in G.

Theorem 4.2 Let G be a classical group, T_H be a idempotent T —norm and $\overline{A} \in IF(G^{\cdot},T_H)$. Then $\overline{A} \in NIF(G^{\cdot},T_H)$ iff for all $x,y \in G$, $\overline{A}(y^{-1} \cdot x \cdot y) = \overline{A}(x)$.

proof By Definition 4.1 and the definition of T_H , its proof is straightforward.

Applying Definition 3.1, 4.1 and Theorem 4.2 to the above space $NIF(\overline{G}^{\cdot},T_H)$, We obtain the following theorem.

Theorem 4.3 Let G and $\overline{G}$ be classical groups, the mapping $f: G \rightarrow \overline{G}$ be a homomorphism in the groups, T_H be a idempotent T —norm. Then

(1)$F_f(\overline{A}) \in NIF(\overline{G}^{\cdot},T_H)$ if $\overline{A} \in NIF(G^{\cdot},T_H)$

(2)$F_f^{-1}(\overline{B}) \in NIF(G^{\cdot},T_H)$ if $\overline{B} \in NIF(\overline{G}^{\cdot},T_H)$

proof We only verify (2). By Definition 3.1, for any $x,y \in G$ we can infer that

$$F_f^{-1}(\overline{B})(x \cdot y) = \overline{B}(f(x \cdot y)) = \overline{B}(f(x) \cdot f(y)) = \overline{B}(f(y) \cdot f(x))$$
$$= \overline{B}(f(y \cdot x)) = F_f^{-1}(\overline{B})(y \cdot x)$$

Consequently, $F_f^{-1}(\overline{B}) \in NIF(G^{\cdot},T_H)$

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