Hypergroup Structures Over Families of Fuzzy Subsets of A Quasi-F-Polygroup

M.M.Zahedi , A.Hasankhani

Department of Mathematics, Kerman University

Kerman, IRAN

Abstract:

Let $(H,*)$ be a quasi-F-Polygroup and let $q : H \leftrightarrow [0,1]_*^H$ be a function, where $[0,1]_*^H$ is the set of all non-zero function of H to $[0,1] \subseteq \mathbb{R}$. We consider the set $H_q = \{q(a) : a \in H\}$, and we answer to the following question:

what are the conditions for the F-Polygroupoid $(H,*)$ and the function q such that (H_q, o) is a semihypergroup (hypergroup)? Where "o" is a function which is defined as follows:

$$\begin{aligned} o : H_q \times H_q &\leftrightarrow P_*(H_q) \\ (q(a), q(b)) &\longmapsto \{q(x) : x \in supp(q(a) * q(b))\}. \end{aligned}$$

Keywords: Hypergroupoid, Semihypergroup, Hypergroup, F-polygroupoid, Semi-F-Polygroup, Quasi-F-Polygroup, F-Polygroup

1. Introduction

Let $(H,.)$ be a hypergroupoid and let $q : H \leftrightarrow P_*(H)$ be a function. On the set $H_q = \{q(a) : a \in H\}$ a function "o" can be defined by:

$$q(a)oq(b) = \{q(c) : c \in q(a).q(b)\},$$

and it is called a hyperoperation on H_q.

This type of hyperoperation has been studied by M. Krasner, P. Lecomte [5], [6], Y. Sureau [9], [10] and M. Gutan [3]. Zadeh in [11] introduced fuzzy subset μ of a non-empty set A, as a function of A to interval $[0,1] \subseteq \mathbb{R}$. In [12], [13], [14], [15] we introduced and studied the notion of fuzzy subpolygroup and F-Polygroup, which these structures generalise the concepts of fuzzy subgroup [8] and polygroup [1].

Now in this paper we extend some results that those obtained by M. Gutan [3], to fuzzy subsets of a quasi-F-Polygroup H.

2. Preliminaries

Throughout this paper I is the unit interval $[0,1] \subseteq \mathbb{R}$, and if $\mu \in I^A$, then by $supp(\mu)$ we mean the set $\{x \in A | \mu(x) \neq 0\}$. Let $\mu, \eta \in I^A$. Then $\mu \leq \eta$ iff $\mu(x) \leq \eta(x)$, For all $x \in A$.

Definition 2.1 [11]. Let μ, η and $\mu_\alpha \in I^A$ where α is in the index set Λ. We define the fuzzy subsets $\mu \cap \eta, \mu \cup \eta, \bigcap_{\alpha\in\Lambda} \mu_\alpha$ and $\bigcup_{\alpha\in\Lambda} \mu_\alpha$ as follows:

(*i*) $(\mu \cap \eta)(x) = \min\{\mu(x), \eta(x)\}$,

(*ii*) $(\mu \cup \eta)(x) = \max\{\mu(x), \eta(x)\}$,

(*iii*) $(\bigcap_{\alpha\in\Lambda} \mu_\alpha)(x) = \inf_{\alpha\in\Lambda} \mu_\alpha(x)$,

(*iv*) $(\bigcup_{\alpha\in\Lambda} \mu_\alpha)(x) = \sup_{\alpha\in\Lambda} \mu_\alpha(x)$, for all $x \in A$.

Definition 2.2. Let $A \neq \emptyset$ and "o" be a function from $A \times A$ to $P^*(A) = P(A)\backslash\{\emptyset\}$. Then "$o$" is called a hyperoperation on A.

Definition 2.3. If $X, Y \in P^*(A)$, then we define XoY as:

$$XoY = \bigcup_{(x,y)\in X\times Y} xoy.$$

Notation. Let "o" be a hyperoperation on A and $a \in A, X \in P^*(A)$. Then by aoX and Xoa we mean $\{a\}oX$ and $Xo\{a\}$ respectively.

Definition 2.4 [7]. Let $H \neq \phi$ and "o" a hyperoperation on H. then (H, o) is called a hypergroupoid and if $xo(yoz) = (xoy)oz$ for all $x, y, z \in H$, then (H, o) is called a semihypergroup.

Definition 2.5 [7]. A semihypergroup (H, o) is called a hypergroup iff

$$xoH = Hox = H, \quad \forall x \in H.$$

Definition 2.6 [1]. Let "o" be a hyperoperation on A. Then (A, o) is called a polygroup iff

(*i*) $xo(yoz) = (xoy)oz \quad \forall\, x, y, z \in A$,

(*ii*) There exists an element $e \in A$ such that

$$xoe = eox = \{x\}, \quad \forall\, x \in A.$$

(e is called the identity element of A.),

(*iii*) for each $x \in A$, there exists a unique element $x' \in A$ such that

$$e \in xox' \cap x'ox.$$

(x' is called the inverse of x and is denoted by x^{-1}.),

(*iv*) $z \in xoy \Rightarrow x \in zoy^{-1} \Rightarrow y \in x^{-1}oz, \forall x, y, z \in A$.

For any subset A of X, we let χ_A denote the characteristic function of A.

Definition 2.7. Let $A \neq \emptyset$ and $I_*^A = I^A\backslash\{o\}$, where o is the function which is identically 0. Then

(*i*) by an F-hyperoperation "$*$" on A we mean a function from $A \times A$ to I_*^A; in other words for any $a, b \in A, a * b$ is a non-empty fuzzy subset of A.

(*ii*) if $\mu, \eta \in I_*^A$, then $\mu * \eta \in I_*^A$ is defined by

$$\mu * \eta = \bigcup_{x\in supp(\mu), y\in supp(\eta)} x * y.$$

Notation 2.8. Let $\mu \in I_*^A$, $B\&C \in P^*(A)$ and $a \in A$. Then

(i) $a * \mu$ and $\mu * a$ denote $\chi_{\{a\}} * \mu$ and $\mu * \chi_{\{a\}}$ respectively,

(ii) $a * B, B * a, \mu * B, B * \mu$ and $B * C$ denote $\chi_{\{a\}} * \chi_B, \chi_B * \chi_{\{a\}}, \mu * \chi_B, \chi_B * \mu$ and $\chi_B * \chi_C$ respectively.

Definition 2.9. Let " $*$ " be an F-hyperoperation on the non-empty set A. Then $(A, *)$ is called an F-polygroupoid.

Lemma 2.10 [13]. Let $(A, *)$ be a F-polygroupoid and $\mu_1, \mu_2 \in I_*^A$. Then

(i) $\bigcup_{x \in supp(\mu_1)} x * \mu_2 = \mu_1 * \mu_2 = \bigcup_{y \in supp(\mu_2)} \mu_1 * y,$

(ii) $t * \mu_2 \le \mu_1 * \mu_2$ and $\mu_2 * t \le \mu_2 * \mu_1$, $\forall t \in supp(\mu_1)$,

(iii) if $\mu_1 \le \mu_2$, then

$$\mu_1 * x \le \mu_2 * x \quad , \quad x * \mu_1 \le x * \mu_2 \quad , \quad \forall x \in A$$

Definition 2.11. Let $(H, *)$ be a F-polygroupoid. Then $(H, *)$ is called a semi-F-polygroup iff

$$x * (y * z) = (x * y) * z, \quad \forall x, y, z \in H.$$

Lemma 2.12 [13]. Let $(H, *)$ be a semi-F-polygroup and $\mu_1, \mu_2, \mu_3 \in I_*^H$. Then

$$(\mu_1 * \mu_2) * \mu_3 = \mu_1 * (\mu_2 * \mu_3).$$

Definition 2.13. Let $\mathcal{F}$ be a non-empty set. Then $(\mathcal{F}, *)$ is called a quasi-F-polygroup iff

(i) $(\mathcal{F}, *)$ is a semi-F-polygroup,

(ii) there exists an element $e \in \mathcal{F}$ such that

$$x \in supp(x * e \cap e * x), \quad \forall x \in \mathcal{F}$$

(In this case we say e is an identity element of $\mathcal{F}$.),

(iii) for each $x \in \mathcal{F}$, there exists a unique element $x' \in \mathcal{F}$ such that

$$e \in supp(x * x' \cap x' * x).$$

(x' is called the inverse of x and it is denoted by x^{-1}.)

Defintion 2.14 [13]. Let $(\mathcal{F}, *)$ be a quasi-F-polygroup. Then $(\mathcal{F}, *)$ is an F-polygroup iff

$$z \in supp(x * y) \Rightarrow x \in supp(z * y^{-1}) \Rightarrow y \in supp(x^{-1} * z), \quad \forall x, y, z \in \mathcal{F}.$$

(This property is called the reversibility of $\mathcal{F}$ with respect to " $*$ ".)

3- Hypergroup structures over families of fuzzy subsets of a quasi-F-polygroup

Let $(H, *)$ be an F-polygroupoid and $q : H \leftrightarrow I_*^H$ a function. We define the functions $\bar{q}$ and $\tilde{q}$ as follows:

$$\bar{q} \; : \; I_*^H \leftrightarrow I_*^H \quad , \quad \bar{q}(\mu) = \bigcup_{a \in supp(\mu)} q(a),$$

$$\tilde{q} \;:\; I_*^H \to P_*(I_*^H)\;,\;\; \tilde{q}(\mu) = \{q(a) : a \in supp(\mu)\}, \text{ for all } \mu \in I_*^H.$$

Clearly, if $a \in H$, then

$$\tilde{q}(\chi_{\{a\}}) = \{q(a)\} \;\;\&\;\; \tilde{q}(\chi_{\{a\}}) = q(a) \tag{1}$$

If $H_q = \tilde{q}(\chi_H)$ $(i.e. H_q = \{q(a) : a \in H\})$, we define the hyperoperation "o" on H_q by:

$$q(a)oq(b) = \tilde{q}(q(a) * q(b)), \quad \forall a, b \in H \tag{2}$$

$(i.e. q(a)oq(b) = \{q(x) : x \in supp(q(a) * q(b))\} \;\; \forall a, b \in H)$ Clearly (H_q, o) is a hypergroupoid.

Definition 3.1. Let $(H, *)$ be an F-polygroupoid and $\mu, \eta \in I_*^H$. Then we say μ and η are equivalent, if $\tilde{q}(\mu) = \tilde{q}(\eta)$. In this case we write: $\mu \equiv \eta$. Clearly the relation $\equiv$ is an equivalence relation over I_*^H.

Lemma 3.2. If $(H, *)$ is an F-polygroupoid and $\mu, \eta \in I_*^H$, then

$$\tilde{q}(\mu)o\tilde{q}(\eta) = \tilde{q}(\tilde{q}(\mu) * \tilde{q}(\eta)).$$

Proof. At first we show that

$$\bigcup_{a\in supp(\mu), b\in supp(\eta)} \tilde{q}(q(a) * q(b)) = \tilde{q}\Big(\bigcup_{a\in supp(\mu), b\in supp(\eta)} (q(a) * q(b))\Big). \tag{3}$$

Let $y \in \bigcup_{a\in supp(\mu), b\in supp(\eta)} \tilde{q}(q(a) * q(b))$ be arbitrary. Then $y \in \tilde{q}(q(a_o) * q(b_o))$ for some $a_o \in supp(\mu)$ and $b_o \in supp(\eta)$. Thus

$$y = q(x), \;\; \exists x \in supp(q(a_o) * q(b_o)).$$

Since

$$q(a_o) * q(b_o) \le \bigcup_{a\in supp(\mu), b\in supp(\eta)} (q(a) * q(b)),$$

we get

$$x \in supp\Big(\bigcup_{a\in supp(\mu), b\in supp(\eta)} (q(a) * q(b)\Big),$$

which implies that

$$y \in \tilde{q}\Big(\bigcup_{a\in supp(\mu), b\in supp(\eta)} (q(a) * q(b)\Big).$$

Therefore

$$\bigcup_{a\in supp(\mu), b\in supp(\eta)} \tilde{q}(q(a) * q(b)) \subseteq \tilde{q}\Big(\bigcup_{a\in supp(\mu), b\in supp(\eta)} (q(a) * q(b))\Big).$$

It is easy to verify that

$$\tilde{q}\Big(\bigcup_{a\in supp(\mu), b\in supp(\eta)} (q(a) * q(b)\Big) \subseteq \bigcup_{a\in supp(\mu), b\in supp(\eta)} \tilde{q}(q(a) * q(b))).$$

Consequently (3) holds.

Now we show that

$$\tilde{q}\Big(\bigcup_{a\in supp(\mu), b\in supp(\eta)} (q(a) * q(b)\Big) = \tilde{q}(\tilde{q}(\mu) * \tilde{q}(\eta)) \tag{4}$$

Let $y \in \bar{q}(\bigcup_{a \in supp(\mu), b \in supp(\eta)} (q(a) * q(b))$ be an arbitrary element.

Then $y = q(x)$ for some $x \in supp(\bigcup_{a \in supp(\mu), b \in supp(\eta)} (q(a) * q(b))$. Therefore there exist $a_o \in supp(\mu)$ and $b_o \in supp(\eta)$, such that $x \in supp(q(a_o) * q(b_o))$. Since

$$q(a_o) * q(b_o) \leq \bigcup_{a \in supp(\mu)} q(a) * \bigcup_{b \in supp(\eta)} q(b),$$

we have

$$y \in \bar{q}(\bigcup_{a \in supp(\mu)} q(a) * \bigcup_{b \in supp(\eta)} q(b)) = \bar{q}(\bar{q}(\mu) * \bar{q}(\eta)).$$

Conversely, let $y \in \bar{q}(\bar{q}(\mu) * \bar{q}(\eta))$ be an arbitrary element. Then $y = q(x)$, for some $x \in supp(\bigcup_{a \in supp(\mu)} q(a) * \bigcup_{b \in supp(\eta)} q(b))$. Thus there exist $t \in supp(\bigcup_{a \in supp(\mu)} q(a))$ and $w \in supp(\bigcup_{b \in supp(\eta)} q(b))$ such that $x \in supp(t * w)$. Hence there are $a_o \in supp(\mu)$ and $b_o \in supp(\eta)$ such that $t \in supp(q(a_o))$ and $w \in supp(q(b_o))$. Thus

$$o < (t * w)(x) \leq (q(a_o) * q(b_o))(x) \leq (\bigcup_{a,b} (q(a) * q(b))(x).$$

Consequently $y \in \bar{q}(\bigcup_{a,b} q(a) * q(b))$. Hence (4) holds.

Finally we show that

$$\bar{q}(\mu) o \bar{q}(\eta) = \bar{q}(\bar{q}(\mu) * \bar{q}(\eta)) \quad (5)$$

$$\begin{aligned} \bar{q}(\mu) o \bar{q}(\eta) &= \bigcup_{a \in supp(\mu), b \in supp(\eta)} q(a) o q(b) \\ &= \bigcup_{a,b} \bar{q}(q(a) * q(b)) \\ &= \bar{q}(\bigcup_{a,b} (q(a) * q(b))), \text{ by (3)} \\ &= \bar{q}(\bar{q}(\mu) * \bar{q}(\eta)), \text{ by (4).} \end{aligned}$$

Consequently (5) holds.

Theorem 3.3. The hypergroupoid (H_q, o) is a semihypergroup if and only if

$$\bar{q}(q(a) * q(b)) * q(c) \equiv q(a) * \bar{q}(q(b) * q(c)) \; \forall a, b, c \in H. \quad (6)$$

Proof. Let $a, b, c \in H$. Then

$$\begin{aligned} q(a) o (q(b) o q(c)) &= \bar{q}(\chi_{\{a\}}) o (\bar{q}(\chi_{\{b\}}) o \bar{q}(\chi_{\{c\}})). \text{ by (1)} \\ &= \bar{q}(\chi_{\{a\}}) o (\bar{q}(\bar{q}(\chi_{\{b\}}) * \bar{q}(\chi_{\{c\}})) \\ &= \bar{q}(\chi_{\{a\}}) o \bar{q}(q(b) * q(c)), \text{ by (1)} \\ &= \bar{q}(\bar{q}(\chi_{\{a\}}) * \bar{q}(q(b) * q(c)), \text{ by Lemma 3.2.} \\ &= \bar{q}(q(a) * \bar{q}(q(b) * q(c))), \text{ by (1).} \end{aligned}$$

Similarly $(q(a) o q(b)) o q(c) = \bar{q}(\bar{q}(q(a) * q(b)) * q(c))$. Consequently (H_q, o) is a semihypergroup if and only if (6) holds.

Corollary 3.4. If $(H, *)$ is a semi-F-polygroup and one of the following conditions holds.

(i) $\bar{q}(q(a) * q(b)) = q(a) * q(b),\ \forall a, b \in H,$

(ii) $q(a) * q(b) \equiv a * b,\ \forall a, b \in H,$

(iii) $q(a) * q(b) = a * b,\quad \forall a, b \in H,$.

then (H_q, o) is a semihypergroup.

Proof. Let (i) hold. Then for any $a, b, c \in H$ we have

$$(q(a) * q(b)) * q(c) = q(a) * q(b) * (q(c)),\quad \text{by Lemma 2.12}$$

On the other hand

$$(q(a) * q(b)) * q(c) = \bar{q}(q(a) * q(b)) * q(c),$$

and

$$q(a) * (q(b) * q(c)) = q(a) * \bar{q}(q(b) * q(c)).$$

Hence (H_q, o) is a semihypergroup, by Theorem 3.3.

Now let (ii) hold. Then:

$$\begin{aligned}
& \bar{q}(\bar{q}(q(a) * q(b)) * q(c)) \\
= & \ \bar{q}(\bar{q}(q(a) * q(b)) * \bar{q}(\chi_{\{c\}})),\ \text{by (1)} \\
= & \ \bar{q}(q(a) * q(b) o \bar{q}(\chi_{\{c\}})\ \text{by Lemma 3.2} \\
= & \ \bar{q}(a * b) o \bar{q}(\chi_{\{c\}}),\ \text{by hypothesis} \\
= & \bigcup_{x \in supp(a*b)} (q(x) o q(c),\ \text{by definition of } \bar{q} \text{ and (1)} \\
= & \bigcup_{x \in supp(a*b)} \bar{q}(q(x) * q(c)),\ \text{by (2)} \\
= & \bigcup_{x \in supp(a*b)} \bar{q}(x * c),\ \text{by hypothesis} \\
= & \ \bar{q}(\bigcup_{x \in supp(a*b)} (x * c)),\ \text{by defintion of } \bar{q} \\
= & \ \bar{q}((a * b) * c).
\end{aligned}$$

Similarly $\bar{q}(q(a) * \bar{q}(q(b) * q(c))) = \bar{q}(a * (b * c))$. Thus (H_q, o) is a semihypergoup, by Theorem 3.3.

If (iii) satisfies, then obviously (ii) holds.

Theorem 3.5. Let $(H, *)$ be a quasi-F-polygroup. If one of the following conditions holds:

(i) $q(a) * q(b) \equiv a * b,\ \forall a, b \in H$

(ii) $q(a) * q(b) = a * b,\ \forall a, b \in H,$

then (H_q, o) is a hypergroup.

Proof. Let (i) hold. Then by Corollary 3.4 (ii), (H_q, o) is a semihypergroup. we shall show that

$$H_q o q(a) = H_q = q(a) o H_q,\ \forall q(a) \in H_q.$$

If $a \in H$, then $a \in supp(a * e)$. Thus, the hypothesis and (2) imply that

$$q(a) \in q(a)oq(e), \quad \forall a \in H.$$

Similarly $q(a) \in q(e)oq(a)$, $\forall a \in H$

Now if $b \in H$, is arbitrary, then we have:

$$\begin{aligned} q(b) \in q(b)oq(e) &\subseteq q(b)o(q(a^{-1})oq(a)) \\ &= (q(b)oq(a^{-1}))oq(a) \\ &= H_q oq(a). \end{aligned}$$

Hence $H_q \subseteq H_q oq(a) \subseteq H_q$. That is $H_q = H_q oq(a)$. Similarly $H_q = q(a)oH_q$.

If (ii) satisfies, then obviously (i) holds.

Theorem 3.6. Let $(H, *)$ be a quasi-F-Polygroup and (6) hold. If

$$a \in supp(q(a)), \forall a \in H, \tag{7}$$

then (H_q, o) is a hypergroup.

Proof. (7) implies that

$$\bar{q}(a * b) \subseteq \bar{q}(q(a) * q(b)) = q(a)oq(b), \forall a, b \in H.$$

Since $a \in supp(a*e \cap e*a)$ and $e \in supp(a*a^{-1} \cap a^{-1}*a)$, we get that, $q(a) \in q(a)oq(e)$, $q(a) \in q(e)oq(a)$, $q(e) \in q(a)oq(a^{-1})$ and $q(e) \in q(a^{-1})oq(a)$ respectively. Now the rest of the proof is similar to that which stated in Theorem 3.5.

Example 3.7. Let $(H, *)$ be a quasi-F-Polygroup such that $supp(e*e) = \{e\}$. We define the function

$$q : H \leftrightarrow I_*^H, q(a) = a * e, \forall a \in H.$$

Then for all $a, b \in H$, we have:

$$\begin{aligned} \bar{q}(q(a) * q(b)) &= \bar{q}(a * e * b * e) \\ &= \bigcup_{x \in supp(a*e*b*e)} q(x) \\ &= \bigcup_{x \in supp(a*e*b*e)} x * e \\ &= a * e * b * e * e \\ &= a * e * b * e \\ &= q(a) * q(b) \end{aligned}$$

Hence, by Corollary 3.4 (i), (H_q, o) is a semihypergroup. Therefore by Theorem 3.3, (6) holds. Now since $a \in supp(q(a))$, then from Theorem 3.6, we conclude that (H_q, o) is a hypergroup.

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