($\in \vee q$)-level subset

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Abstract

The notion of ($\in \vee q$)-level subset is introduced. The study of ($\in \vee q$)-level subsets of an ($\in, \in \vee q$)-fuzzy subgroup (subring or ideal) are dealt with.

Keywords: *Fuzzy algebra, level subset, ($\in \vee q$)-level subset, fuzzy subgroup. ($\in, \in \vee q$)-fuzzy subgroup, ($\in, \in \vee q$)-fuzzy normal subgroup, ($\in, \in \vee q$)-fuzzy ideal, ($\in, \in \vee q$)-fuzzy radical.*

1. Introduction.

Fuzzy subgroup was introduced by Rosenfeld [6]. Liu [4] introduced the notion of fuzzy subring and ideal. Since then different researchers have contributed significantly for the development of these literature. Using the notions of "belongingness ($\in$)" and "quasi-coincidence (q)" of fuzzy points with fuzzy sets the concept of (α, β)-*fuzzy subgroup* where α, β are any two of $\{ \in, q, \in \vee q, \in \wedge q \}$ with $\alpha \neq \in \wedge q$ is introduced in [1]. It was found that the most viable generalisation of Rosenfeld's fuzzy subgroup is the notion of ($\in, \in \vee q$)-*fuzzy subgroup*. The detailed study with ($\in, \in \vee q$)-fuzzy subgroup has been considered in [2]. It was found that a fuzzy subset λ of a group G is an ($\in, \in \vee q$)-fuzzy subgroup of G if and only if $\lambda_t = \{x \in G;\ \lambda(x) \geq t\}$ is a subgroup of G $\forall\ 0 < t \leq 0.5$. On the other hand a fuzzy subset λ of a group G is a Rosenfeld's fuzzy subgroup (or ($\in, \in$)-fuzzy subgroup) if and only if λ_t is a subgroup $\forall\ t \in (0, 1]$. Similar type of asymmetry for the values of t occured in the cases of ($\in, \in \vee q$)- fuzzy subrings, ideals. Now it would be an interesting study of level subsets of different fuzzy subsystems λ when "$\lambda(x) \geq t$ or $\lambda(x) + t > 1$" or equvalently "$x_t \in \vee q\ \lambda$". With this object, the notion of a new type of level subset λ_t of a non-empty set G, called ($\in \vee q$)-**level subset** is introduced where $\lambda_t = \{x \in G;\ \lambda(x) \geq t$ or $\lambda(x) + t > 1\} = \{x \in G;\ x_t \in \vee q\ \lambda\}$.

The most significant achievement of this study is that for any fuzzy subset λ of X (group or ring)is an ($\in, \vee q$)-fuzzy subgroup (or subring or ideal) of X if and only if the ($\in \vee q$)-level subset $\lambda_t = \{x \in X;\ x_t \in \vee q\ \lambda\}$ is a subgroup (or subring or ideal) of X. The results with level subsets obtained in [2], [3] are verified with ($\in \vee q$)-level subset.

2. Preliminaries.

Let G be a non-empty set.

Definition 2.1 [Zadeh] [7]. A map $\lambda : G \to [0, 1]$ is called a ***fuzzy subset*** of G.

Definition 2.2 [Ming and Ming] [5]. A fuzzy subset λ of G of the form

$$\lambda(y) = \begin{cases} t(\neq 0) & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a **fuzzy point** with **support** x and value t and is denoted by x_t.

Definition 2.3 [Ming and Ming] [5]. A fuzzy point x_t is said to **belong to** (resp. be **quasi-coincident** with) a fuzzy set λ, written as $x_t \in \lambda$ (resp. $x_t\ q\ \lambda$) if $\lambda(x) \geq t$(resp.$\lambda(x) + t > 1$).

"$x_t \in \lambda$ or $x_t\ q\ \lambda$" will be denoted by $x_t \in \vee q\ \lambda$.

Definition 2.4. Let λ be a fuzzy subset of G. Then $\forall t \in (0, 1]$, the set $\lambda_t = \{x \in G;\ \lambda(x) \geq t\}$ is called **level subset** of λ.

Definition 2.5. A fuzzy subset λ of G is said to have the "**sup property**" if for any non-empty subset T of R, there exists $a \in T$ such that

$\lambda(a) = sup\{\lambda(t);\ t \in T\}$.

Let G be a group.

Definition 2.6 [Bhakat and Das] [1]. A fuzzy subset λ of G is said to be an $(\in, \in \vee\, q)$-**fuzzy subgroup** of G if $\forall x, y \in G$ and $t, r \in (0, 1]$

(i) $x_t \in \lambda, y_r \in \lambda \Rightarrow (xy)_{M(t,r)} \in \vee\, q\ \lambda$

(ii) $x_t \in \lambda \Rightarrow (x^{-1})_t \in \vee\, q\ \lambda$.

Theorem 2.7 [Bhakat and Das] [2]. (i) A necessary and sufficient condition for a fuzzy subset λ of a group G to be an $(\in, \in \vee\, q)$-fuzzy subgroup of G is $\lambda(xy^{-1}) \geq M(\lambda(x), \lambda(y), 0.5) \quad \forall x, y \in G$.

(ii) Let λ be a fuzzy subgroup of G. Then $\lambda_t = \{x \in G;\ \lambda(x) \geq t\}$ is a subgroup of G $\quad \forall 0 < t \leq 0.5$. Conversely, if λ is a fuzzy subset of G such that λ_t is a subgroup of G $\quad \forall t \in (0, 0.5]$, then λ is an $(\in, \in \vee\, q)$-fuzzy subgroup of G.

(iii) Let G be a group. Then given any chain of subgroups $G_0 \subset G_1 \subset .... \subset G_r = G$, there exists a fuzzy subgroup of G whose level subgroups are precisely the members of the chain.

Definition 2.8 Bhakat and Das] [2]. An $(\in, \in \vee\, q)$-fuzzy subgroup λ of G is said to be $(\in, \in \vee\, q)$-fuzzy normal if for any $x, y \in G$ and $t \in (0, 1]$,

$x_t \in \lambda \Rightarrow (yxy^{-1})_t \in \vee\, q\ \lambda$.

Theorem 2.9 [Bhakat and Das] [2]. Let λ be an $(\in, \in \vee\, q)$-fuzzy normal subgroup of G. Then $\lambda_t = \{x \in G;\ \lambda(x) \geq t\}$ is a normal subgroup of G $\quad \forall 0 < t \leq 0.5$. Conversely, if λ is a fuzzy subset of G such that λ_t is normal subgroup of G $\quad \forall 0 < t \leq 0.5$, then λ is $(\in, \in \vee\, q)$-fuzzy normal.

Definition 2.10 [Bhakat and Das] [3]. A fuzzy subset λ of a ring R is said to be an $(\in, \in \vee\, q)$-fuzzy subring of R if $\quad \forall x, y \in R$ and $t, r \in (0, 1]$,

$(i) x_t, y_r \in \lambda \Rightarrow (x + y)_{M(t,r)} \in \vee\, q\ \lambda$,

(ii) $x_t \in \lambda \Rightarrow (-x)_t \in \vee\, q\ \lambda$

(iii) $x_t, y_r \in \lambda \Rightarrow (xy)_{M(t,r)} \in \vee\, q\ \lambda$.

Theorem 2.11 [Bhakat and Das] [3]. A fuzzy subset λ of a ring R is an $(\in, \in \vee\, q)$-fuzzy subring of R if and only if $\lambda(x - y), \lambda(xy) \geq M(\lambda(x), \lambda(y), 0.5) \quad \forall x, y \in R$.

Definition 2.12 [Bhakat and Das] [3]. A fuzzy subset λ of a ring R is said to be $(\in, \in \vee\, q)$-fuzzy ideal of R if

(i) λ is an $(\in, \in \vee\, q)$-fuzzy subring of R,

(ii) $x_t \in \lambda$ and $y \in R \Rightarrow (xy)_t, (yx)_t \in \vee\, q\ \lambda$.

Theorem 2.13 [Bhakat and Das] [3]. A fuzzy subset λ of a ring R is an $(\in, \in \vee\, q)$-fuzzy ideal of R if and only if

(i) $\lambda(x - y) \geq M(\lambda(x), \lambda(y), 0.5)$,

(ii) $\lambda(xy), \lambda(yx) \geq M(\lambda(x), 0.5) \quad \forall x, y \in R$.

Definition 2.14 [Bhakat and Das] [3]. An $(\in, \in \vee\, q)$-fuzzy ideal of R is said to be

(i) $(\in, \in \vee\, q)$-fuzzy semiprime, if $\quad \forall x, y \in R$ and $t \in (0, 1], (x^2)_t \in \lambda \Rightarrow x_t \in \vee\, q\ \lambda$,

(ii) $(\in, \in \vee\, q)$-fuzzy prime, if $\forall x, y \in R$ and $t \in (0, 1]$, $(xy)_t \in \lambda \Rightarrow x_t \in \vee\, q\ \lambda$ or $y_t \in \vee\, q\ \lambda$.

(iii) $(\in, \in \vee\, q)$-fuzzy semiprimary, if $\forall x, y \in R$ and $t \in (0, 1]$, $(xy)_t \in \lambda \Rightarrow x_t^n \in \vee\, q\ \lambda$ or $y_t^m \in \vee\, q\ \lambda$ for some $n, m \in N$,

(iv) $(\in, \in \vee\, q)$-fuzzy semiprimary, if $\forall x, y \in R$ and $t \in (0, 1]$, $(xy)_t \in \lambda \Rightarrow x_t \in \vee\, q\ \lambda$ or $y_t^n \in \vee\, q\ \lambda$ for some $n \in N$.

Theorem 2.15 [Bhakat and Das] [3]. A fuzzy subset λ of a ring R is an $(\in, \in \vee\, q)$-fuzzy subring (ideal) of R if and only if λ_t is a subring (ideal) of R $\quad \forall t \in (0, 0.5]$.

Theorem 2.16 [Bhakat and Das] [3]. An $(\in, \in \vee\, q)$-fuzzy ideal of R is $(\in, \in \vee\, q)$-fuzzy prime if and only if $Max\{\lambda(x), \lambda(y)\} \geq M(\lambda(xy), 0.5) \quad \forall x, y \in R$.

Theorem 2.17 [Bhakat and Das] [3]. A fuzzy ideal λ of a ring R is an $(\in, \in \vee\, q)$-fuzzy semiprime (or prime or semiprimary

or primary) if and only if λ_t is semiprime (or prime or semiprimary or primary) $\forall 0 < t \leq 0.5$.

Definition 2.18 [Bhakat and Das] [3]. Let λ be an $(\in, \in \vee q)$-fuzzy ideal of R. The fuzzy subset $Rad\lambda$ of R defined by

$$(Rad\lambda)(x) = \begin{cases} M(sup\{\lambda(x^n);\ n \in N\}, 0.5) & \text{if } \lambda(x) < 0.5 \\ \lambda(x) & \text{if } \lambda(x) \geq 0.5 \end{cases}$$

is called the $(\in, \in \vee q)$-fuzzy radical of λ.

Theorem 2.19 [Bhakat and Das] [3]. Let λ be an $(\in, \in \vee q)$-fuzzy ideal of R. Then
(i) $Rad\lambda$ is an $(\in, \in \vee q)$-fuzzy ideal of R.
(ii) If λ has the "sup property", then $\forall 0 < t \leq 0.5$, $Rad\lambda_t = (Rad\lambda)_t$.

3. $(\in \vee q)$- level subset.

Let X be a non-empty set and I denote the closed unit interval [0,1]. $t, r \in (0,1]$. λ, μ will denote any fuzzy subset of X.

Definition 3.1. *The subset $\lambda_t = \{x \in X;\ \lambda(x) \geq t$ or $\lambda(x) + t > 1\} = \{x \in X;\ x_t \in \vee q\ \lambda\}$ is called* **$(\in \vee q)$-level subset** *of X.*

Remark 3.2. It follows from the definition of $(\in \vee q)$-level subset that $\{x \in X;\ \lambda(x) \geq t\} \subseteq \{x \in X;\ x_t \in \vee q\ \lambda\}$. However, the reverse set inclusion relation may not be true.

Example 3.3. Let $X = \{a,\ b,\ c\}$ and $\lambda = (0.6,\ 0.3,\ 0.8)$. Then $\{x \in X;\ x_{0.7} \in \vee q\ \lambda\} = \lambda_{0.7} = \{a,\ c\} \neq \{c\} = \{x \in X;\ \lambda(x) \geq 0.7\}$.

Theorem 3.4.
(i) $(\lambda \cup \mu)_t = \lambda_t \cup \mu_t$
(ii) $(\lambda \cap \mu)_t = \lambda_t \cap \mu_t$.

Proof. (i) $x \in (\lambda \cup \mu)_t \Leftrightarrow x_t \in \vee q\ (\lambda \cup \mu) \Leftrightarrow (\lambda \cup \mu)(x) \geq t$ or $(\lambda \cup \mu)(x) + t > 1 \Leftrightarrow \{\lambda(x) \geq t$ or $\mu(x) \geq t\}$ or $\{\lambda(x) + t > 1$ or $\mu(x) + t > 1\} \Leftrightarrow \{\lambda(x) \geq t$ or $\lambda(x) + t > 1\}$ or $\{\mu(x) \geq t$ or $\mu(x) + t > 1\} \Leftrightarrow x \in \lambda_t$ or $x \in \mu_t \Leftrightarrow x \in (\lambda_t \cup \mu_t)$.

(ii) $x \in (\lambda \cap \mu)_t \Leftrightarrow (\lambda \cap \mu)(x) \geq t$ or $(\lambda \cap \mu)(x) + t > 1 \Leftrightarrow \{\lambda(x) \geq t$ and $\mu(x) \geq t\}$ or $\{\lambda(x) + t > 1$ and $\mu(x) + t > 1\} \Leftrightarrow \{\lambda(x) \geq t$ or $\lambda(x) + t > 1\}$ and $\{\mu(x) \geq t$ or $\mu(x) + t > 1\} \Leftrightarrow x \in \lambda_t$ and $x \in \mu_t \Leftrightarrow x \in (\lambda_t \cap \mu_t)$.

Theorem 3.5.
(i) $\{\lambda \cup (\mu \cap \nu)\}_t = (\lambda \cup \mu)_t \cap (\lambda \cup \nu)_t$
(ii) $\{\lambda \cap (\mu \cup \nu)\}_t = (\lambda \cap \mu)_t \cup (\lambda \cap \nu)_t$

Proof. (i) $\{\lambda \cup (\mu \cap \nu)\}_t = \lambda_t \cup (\mu \cap \nu)_t$ [by Theorem 3.4(i)] $= \lambda_t \cup (\mu_t \cap \nu_t)$ [by Theorem 3.4(ii)] $= (\lambda_t \cup \mu_t) \cap (\lambda_t \cup \nu_t) = (\lambda \cup \mu)_t \cap (\lambda \cup \nu)_t$.
(ii) Similar to (i).

Remark 3.6. If $t > r$, then λ_t may not be a subset of λ_r.

Example 3.7. Let $X = \{a,\ b,\ c\}$ and $\lambda = (0.6,\ 0.2,\ 0.3)$. Then $b \in \lambda_{0.9}$ but $b \notin \lambda_{0.8}$.

Theorem 3.8. $\overline{(\lambda_t)} \subset \{(\overline{\lambda}_t) \cap (\overline{\lambda})_{1-t}\}$ *where* $\overline{\lambda}$ *denote the complement of* λ.

Proof. $x \in \overline{(\lambda_t)} \Rightarrow x \notin \lambda_t \Rightarrow x_t \overline{\in \vee q}\ \lambda \Rightarrow \lambda(x) < t$ and $\lambda(x) + t \leq 1 \Rightarrow -\lambda(x) > -t$ and $-\lambda(x) - t \geq -1 \Rightarrow 1 - \lambda(x) > 1 - t$ and $1 - \lambda(x) - t \geq 0 \Rightarrow \overline{\lambda}(x) > 1 - t$ and $\overline{\lambda}(x) - t \geq 0 \Rightarrow \overline{\lambda}(x) > 1 - t$ and $\overline{\lambda}(x) \geq t \Rightarrow x \in (\overline{\lambda})_{1-t}$ and $x \in (\overline{\lambda})_t \Rightarrow x \in \{(\overline{\lambda})_{1-t} \cap (\overline{\lambda})_t\}$.

Corollary 3.9. $\overline{(\lambda_t)} \subset \{(\overline{\lambda})_t \cup (\overline{\lambda})_{1-t}\}$

Example 3.10. Let $X = \{a,\ b,\ c\}$ and $\lambda = (0.2,\ 0.6,\ 0.4)$. Then $\overline{\lambda} = (0.8,\ 0.4,\ 0.6)$ and $a \in \{(\overline{\lambda})_{0.2} \cap (\overline{\lambda})_{0.3}\} \subset \{(\overline{\lambda})_{0.2} \cup (\overline{\lambda})_{0.3}\}$ but $a \notin \overline{(\lambda_{0.2})}$.

Theorem 3.11. $\overline{(\lambda_t \cup \mu_t)} \subset (\overline{\lambda})_{1-t} \cap (\overline{\mu})_{1-t}$

Proof. $\overline{(\lambda_t \cup \mu_t)} = \overline{(\lambda \cup \mu)_t} \subset (\overline{\lambda})_{1-t} \cap (\overline{\mu})_{1-t}$ [by Theorem 3.8] $= (\overline{\lambda} \cap \overline{\mu})_{1-t} = (\overline{\lambda})_{1-t} \cap (\overline{\mu})_{1-t}$.

Theorem 3.12. *If $\lambda \circ \mu$ has the "sup property", then $(\lambda \circ \mu)_t = \lambda_t.\mu_t$.*

Proof. Let $z \in X$. Then $z \in (\lambda \circ \mu)_t \Rightarrow (\lambda \circ \mu)(z) \geq t$ or $(\lambda \circ \mu)(z) > 1 - t$. Since $\lambda \circ \mu$ has the "sup property" there exist $x_0, y_0 \in X$ such that $z = x_0 y_0$ and $sup\{M(\lambda(x), \lambda(y));\ z = xy\} = M(\lambda(x_0), \mu(y_0))$.

Case I.
$(\lambda o \mu)(z) \geq t \Rightarrow sup\{M(\lambda(x), \mu(y));\ z = xy\} \geq t$.
i.e, $x_0 \in \lambda_t$ and $y_0 \in \mu_t$ and hence $z \in \lambda_t.\mu_t$. So $(\lambda \circ \mu)_t \subset \lambda_t.\mu_t$.

Case II. $(\lambda \circ \mu)(z) > 1 - t$.

Then $M(\lambda(x_0), \mu(y_0)) > 1 - t$, i.e, $x_0 \in \lambda_t$ and $y_0 \in \mu_t$ and thus $z \in \lambda_t.\mu_t$. So $(\lambda \circ \mu)_t \subset \lambda_t.\mu_t$. Again $z \in \lambda_t.\mu_t \Rightarrow \exists\ x, y \in X$ such that $z = xy$ and $x \in \lambda_t$ and $y \in \mu_t \Rightarrow \{\lambda(x) \geq t$ or $\lambda(x) > 1 - t\}$ and $\{\lambda(y) \geq t$ or $\lambda(y) > 1 - t\} \Rightarrow sup\{M(\lambda(x), \mu(y));\ z = xy\} \geq t$ or $sup\{M(\lambda(x), \mu(y));\ z = xy\} > 1 - t \Rightarrow z \in (\lambda \circ \mu)_t$. So $\lambda_t.\mu_t \subset (\lambda o \mu)_t$. Therefore $(\lambda o \mu)_t = \lambda_t.\mu_t$.

Henceforth, unless otherwise mentioned, a fuzzy subgroup or a fuzzy subring will indicate an $(\in, \in \vee q)$-fuzzy subgroup or an $(\in, \in \vee q)$- fuzzy subring. G will denote a group with e as identity and R will denote a ring with 0 as null element.

Theorem 3.13. *A fuzzy subset λ of G is a fuzzy subgroup of G if and only if λ_t is a subgroup for all $t \in (0, 1]$.*

Proof. Let λ be a fuzzy subgroup of G. Let $x, y \in \lambda_t$. Then $\lambda(x) \geq t$ or $\lambda(x) + t > 1$ and $\lambda(y) \geq t$ or $\lambda(y) + t > 1$. Now since $\lambda(xy^{-1}) \geq M(\lambda(x), \lambda(y), 0.5)$ [since λ is a fuzzy subgroup of G], it follows that $\lambda(xy^{-1}) \geq M(t, 0.5)$. For otherwise, $\lambda(xy^{-1}) < M(t, 0.5) \Rightarrow x_t \overline{\in \vee q}\ \lambda$ or $y_t \overline{\in \vee q}\ \lambda$, a contradiction. If $M(t, 0.5) = t$, then $xy^{-1} \in \lambda_t$ and if $M(t, 0.5) = 0.5$, then $\lambda(xy^{-1}) + t > 1$ and hence $xy^{-1} \in \lambda_t$. So λ_t is a subgroup of G. Conversely, let λ be a fuzzy subset of G such that λ_t is a subgroup of G $\quad \forall t \in (0, 1]$. If possible, let $\lambda(xy^{-1}) < t < M(\lambda(x), \lambda(y), 0.5)$ for some $t \in (0, 0.5)$. Then $x, y \in \lambda_t$ and thus $xy^{-1} \in \lambda_t$, i.e, $\lambda(xy^{-1}) \geq t$ or $\lambda(xy^{-1}) + t > 1$, thus in any case a contradiction. Therefore $\lambda(xy^{-1}) \geq M(\lambda(x), \lambda(y), 0.5) \quad \forall x, y \in G$, so λ is a fuzzy subgroup of G.

Theorem 3.14. *Let G be a group. then given any chain of subgroups $G_0 \subset G_1 \subset \ldots\ldots \subset G_r = G$, there exists a fuzzy subgroup of G whose $(\in \vee q)$-level subgroups are precisely the members of the chain.*

Proof. Let $\{t_i;\ t_i \in (0, 0.5);\ i = 1, 2, ..., r\}$ be such that $t_1 > t_2 > \ldots.. > t_r$. Let $\lambda : G \to I$ be defined as follows:

$$\lambda(x) = \begin{cases} t(> 0.5) & \text{if } x = e, \\ u(> t) & \text{if } x \in G_0 - \{e\}, \\ t_1 & \text{if } x \in G_1 - G_0, \\ t_2 & \text{if } x \in G_2 - G_1, \\ \vdots & \\ t_r & \text{if } x \in G_r - G_{r-1} \end{cases}$$

Then λ is a fuzzy subgroup of G. Note that, $\lambda_{0.5} = G_0$ and $\lambda_{t_i} = G_i$ for $i = 1, 2, \ldots., r$. This follows from the fact that $x_t \in \vee\ q\ \lambda \Rightarrow x_t \in \lambda$ if $t \in (0, 0.5)$.

Remark 3.15. If $t_i \notin (0, 0.5)$, then all the members of the chain may not be characterised by the $(\in \vee q)$-level subgroups of λ.

Example 3.16. Let G = Additive group of all integers. Let nG = Additive group of all integers multiple of n. Then $16G \subset 8G \subset 4G \subset 2G \subset G$ be a chain of subgroups of G. Let $\lambda : G \to I$ be defined as follows:

$$\lambda(x) = \begin{cases} 0.6 & \text{if } x = 0 \\ 0.9 & \text{if } x \neq 0, x \in 16G \\ 0.7 & \text{if } x \in 8G - 16G \\ 0.5 & \text{if } x \in 4G - 8G \\ 0.2 & \text{if } x \in 2G - 4G \\ 0.1 & \text{if } x \in G - 2G \end{cases}$$

Then λ is a fuzzy subgroup of G. Note that $\lambda_{0.5} = 4G = \lambda_{0.6} = \lambda_{0.7}$, $\lambda_{0.9} = 2G = \lambda_{0.2}$, $\lambda_{0.1} = G$.

Theorem 3.17. *A fuzzy subset λ is an $(\in, \in \vee q)$-fuzzy normal subgroup of G if and only if λ_t is a normal subgroup of G $\quad \forall t \in (0, 1]$.*

Proof. Let λ be an $(\in, \in \vee q)$-fuzzy normal subgroup of G. Then λ_t is a subgroup of G [by Theorem 3.13]. Let $x \in \lambda_t$. Then $x_t \in \vee\ q\ \lambda$. Now $\lambda(yxy^{-1}) \geq M(\lambda(x), 0.5) \quad \forall y \in G$ [since λ is $(\in, \in \vee q)$-fuzzy normal.] and thus $yxy^{-1} \in \lambda_t$, $\quad \forall y \in G$. So λ_t is normal. Conversely, let λ be a fuzzy subset a fuzzy subset of G such that λ_t is a normal subgroup of G $\forall t \in (0,1]$. Then λ is a fuzzy subgroup of G [by Theorem 3.13]. If possible, let $\lambda(yxy^{-1}) < M(\lambda(x), 0.5)$ for some $x, y \in G$. Choose t such that $\lambda(yxy^{-1}) < t < M(\lambda(x), 0.5)$. Then $x \in \lambda_t$ and thus $yxy^{-1} \in \lambda_t \quad \forall y \in G$, since λ_t is normal, i.e, $yxy^{-1} \in \vee\, q\, \lambda$, i.e, $\lambda(yxy^{-1}) \geq t$ or $\lambda(yxy^{-1} + t > 1$, and thus in any case a contradiction. So $\lambda(yxy^{-1}) \geq M(\lambda(x), 0.5)$ and hence λ is $(\in, \in \vee\, q)$-fuzzy normal.

Theorem 3.18. *A fuzzy subset λ of a ring R is a fuzzy subring (ideal) of R if and only if λ_t is a subring (ideal) of R $\quad \forall t \in (0,1]$.*

Proof. Similar to the proof of Theorem 3.11.

Theorem 3.19. *A fuzzy ideal λ of a ring R is $(\in, \in \vee\, q)$-fuzzy semiprimary (or primary or semiprime or prime) if and only if λ_t is semiprimary (or primary or semiprime or prime) $\quad \forall t \in (0,1]$.*

Proof. Let λ be an $(\in, \in \vee\, q)$-fuzzy prime ideal of R. Then λ_t is an ideal of R $\quad \forall t \in (0,1]$ [by Theorem 3.18]. Let $x, y \in \lambda_t$. Then $(xy)_t \in \vee\, q\, \lambda$. Since λ is an $(\in, \in \vee\, q)$-fuzzy prime ideal of R, $max\{\lambda(x), \lambda(y)\} \geq M(\lambda(xy), 0.5)$. If $(xy)_t \in \lambda$, then either $x_t \in \vee\, q\, \lambda$ or $y_t \in \vee\, q\, \lambda$. If $(xy)_t\, q\, \lambda$, then either $x_t\, q\, \lambda$ or $y_t\, q\, \lambda$. Thus in any case $x \in \lambda_t$ or $y \in \lambda_t$. So λ_t is prime. Conversely, let λ be a fuzzy subset of R such that λ_t is a prime ideal of R $\quad \forall t \in (0,1]$. Then λ is a fuzzy ideal of R [by Theorem 3.18]. Now $(xy)_t \in \lambda \Rightarrow xy \in \lambda_t \Rightarrow x \in \lambda_t$ or $y \in \lambda_t \Rightarrow x_t \in \vee\, q\, \lambda$ or $y_t \in \vee\, q\, \lambda$. Therefore λ is an $(\in, \in \vee\, q)$-fuzzy prime. The proof of the other cases may similarly be disposed of.

Theorem 3.20. *If λ is a fuzzy ideal of R with "sup property', then $\quad \forall t \in (0,1]$, $Rad\lambda_t = (Rad\lambda)_t$.*

Proof. $x \in Rad\lambda_t \Rightarrow x^n \in \lambda_t$ for some $n \in N \Rightarrow (x^n)_t \in \vee\, q\, \lambda \Rightarrow x_t \in \vee\, qRad\lambda \Rightarrow x \in (Rad\lambda)_t$ [since $(Rad\lambda)(x) \geq \lambda(x)$ and $\lambda(x^n) \geq M(\lambda(x), 0.5) \quad \forall n \in N$]. So $Rad\lambda_t \subseteq (Rad\lambda)_t$. Next, let $x \in (Rad\lambda)_t$. Then $x_t \in \vee\, q(Rad\lambda)$. i.e, either $(Rad\lambda)(x) \geq t$ or $(Rad\lambda)(x) + t > 1$.

Case I: Let $(Rad\lambda)(x) \geq t$.

Let $t \leq 0.5$. If $\lambda(x) \leq 0.5$, then $(Rad\lambda)(x) = M(\lambda(x^r), 0.5) \geq t$ for some $r \in N$ [since λ has the "sup property"]. i.e, $x^r_t \in \lambda$ and thus $x^r \in \lambda_t$. Hence $x \in Rad\lambda_t$. If $\lambda(x) > 0.5$, then $(Rad\lambda)(x) = \lambda(x) \geq t$ and hence $x \in Rad\lambda_t$. Next, let $t > 0.5$. Then $(Rad\lambda)(x) = \lambda(x) \geq t$ and thus $x \in Rad\lambda_t$.

Case II: Let $(Rad\lambda)(x) + t > 1$.

So either $M(\lambda(x^r), 0.5) + t > 1$ for some $r \in N$ or $\lambda(x) + t > 1$. i.e, $x^r_t q\, \lambda$ or $x_t\, q\, \lambda$. i.e, $x^r \in \lambda_t$ or $x \in \lambda_t$ i.e, $x \in Rad\lambda_t$. Thus in any case $x \in Rad\lambda_t$. Hence $(Rad\lambda)_t \subseteq Rad\lambda_t$. So $Rad\lambda_t = (Rad\lambda)_t \quad \forall t \in (0,1]$.

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