PRODUCTS OF $(a,b\ ;\ \in_{\alpha}\ ,\ \in_{\alpha} \vee q_{(\alpha,b)})$ - FUZZY SUBGROUPS

Pratyayananda Das
Department of Mathematics
Burdwan University
Golapbag, Burdwan
West Bengal
INDIA

Abstract : Internal direct product , semidirect product , subdirect product and amalgameted product of $(a,b\ ;\in_{\alpha}\ ,\ \in_{\alpha} \vee q_{(\alpha,b)})$- fuzzy subgroups are discussed . Some results concerning level subgroups are obtained .

Keywords : Fuzzy algebra , Fuzzy subgroup , Internal direct product , Subdirect product , Amalgameted product .

1. INTRODUCTION:

Given a subinterval [a,b] of I , with a view to study fuzzy subsystems λ of a universal algebra G such that λ_t are subalgebras of G $\forall\ t \in [a,b]$, the notion of an $(a,b\ ;\ \alpha\ ,\ \beta\ )$ fuzzy subalgebra was introduced in [1] . It was found that $(a,b\ ;\ \in_{\alpha}\ ,\ \in_{\alpha} \vee q_{(\alpha,b)})$- fuzzy subalgebras may play an important role as a useful non-trivial generalisation of Rosenfeld - type fuzzy subalgebras .But to arrive at a definite conclusion , it is necessary to study different aspects of $(a,b\ ;\ \in_{\alpha}\ ,\ \in_{\alpha} \vee q_{(\alpha,b)})$-fuzzy subsystems of different algebraic systems .

Various types of products of fuzzy subgroups (in Rosenfeld's sense) were studied by different authors . In particular , direct product was discussed by Sherwood [4] , internal direct product by Makamba [2] , and semidirect product by Wetherilt [5] . The object of this paper is to study different type of products of $(a,b\ ;\ \in_a\ ,\ \in_a \vee q_{(a,b)})$-fuzzy subgroups , henceforth called only fuzzy subgroups.

Internal direct product and semidirect product are discussed in Section 3 . Subdirect product and amalgameted product are two important types of product in classical group theory . Their fuzzy analogoues are discussed in Sections 4 and 5 respectively . To introduce the notion of subdirect product fuzzy proper functions are used to define a new type of epimorphism of fuzzy subgroups . Some results concerning level subgroups are obtained . It is proved that if λ_i , μ_i , γ_i , δ_i are fuzzy subgroups of a finite group G such that λ_i is the direct product of μ_i and γ_i with amalgameted fuzzy subgroup δ_i for i = 1,2 and if μ_1 be epimorphic to μ_2 , γ_1 be epimorphic to γ_2 under epimorphisms satisfying certain conditions , then λ_1 is epimorphic to λ_2 .

2 . PRELIMINARIES:

Let X be any set and let $I = [0,1]$. I^X will denote the set of all functions $\alpha : X \longrightarrow I$. If $x,y \in I$, $M(x,y)$ will denote the minimum of x and y.

Let $a,b \in I$ be such that $0 < a < b \le 1$.

Let $c = M(2b,1)$, $d = M(2b,1+a)$ and $k = d/2$.

DEFINITION 2.1: Let $\lambda \in I^X$. If $0 < r \le 1$, $0 \le t < 1$ and $0 < \alpha < \beta \le 1$, then λ_r , λ_{st} and $\lambda_{\alpha,\beta}$ are defined by

$$\lambda_r = \{ x \in G;\ \lambda(x) \ge r \}.$$

$\lambda_{st} = \{ x \in G ; \lambda(x) > t \}$.

$\lambda_{\alpha,\beta} = \{ x \in G ; \alpha < \lambda(x) \le \beta \}$.

DEFINITION 2.2: Let X,Y be two sets and let $\lambda \in I^X$, $\mu \in I^Y$. A fuzzy subset F of X×Y is called a proper function from λ to μ if (i) $F(x,y) \le M(\lambda(x), \mu(y))$ $\forall x \in X$, $\forall y \in Y$,

(ii) for each $x \in X$, $\exists y_o \in Y$ such that $F(x,y_o) = \lambda(x)$ and $F(x,y) = \emptyset$ $\forall y \ne y_o$.

A proper function F from λ to μ is said to be (a,b)-surjective if for all $y \in \mu_{sa}$, there exists $x \in \lambda_{sa}$ such that $F(x,y) = \lambda(x)$ and $\lambda(x) \ge \mu(y)$ if $y \in \mu_{a,k}$ and $\lambda(x) \ge k$ if $\mu(y) > k$.

DEFINITION 2.3[1]: Let $\lambda \in I^X$. A fuzzy point (x,t) is said to belong to λ with respect to a , denoted by $(x,t) \in_a \lambda$ (resp.,coincident with λ with respect to (a,b) denoted by $(x,t)\, q_{\langle a,b\rangle}\lambda$) if $\lambda(x) \ge \max\{a,t\}$ (resp. , $\lambda(x) + t > d$).

If $(x,t) \in_a \lambda$ or $(x,t)\, q_{\langle a,b\rangle} \lambda$ then we write $(x,t) \in_a \vee\, q_{\langle a,b\rangle} \lambda$

Let G be a group .

DEFINITION 2.4[1]: A fuzzy subset λ of G is said to be an $(a,b ,\in_a, \in_a \vee q_{\langle a,b\rangle})$ fuzzy subgroup of G if

(i) $(x,t) \in_a \lambda$, $(y,t_1) \in_a \lambda \Rightarrow (xy, M(t,t_1)) \in_a \vee\, q_{\langle a,b\rangle} \lambda$,

(ii) $(x,t) \in_a \lambda \Rightarrow (x^{-1}, t) \in_a \vee\, q_{\langle a,b\rangle} \lambda$

for all $x,y \in G$ and for all $t,t_1 \in (a,c]$.

The conditions (i) and (ii) of Definition 2.4 are equivalent to respectively

(I) $\lambda(xy) \ge M (\lambda(x),\lambda(y), k)$ $\forall x,y \in \lambda_{sa}$,

(II) $\lambda(x^{-1}) \ge M (\lambda(x),k)$ $\forall x \in \lambda_{sa}$.

In what follows , unless otherwise mentioned , by a fuzzy subgroup of G , we shall mean an (a,b $,\in_{a},\in_{a}\vee q_{(a,b)}$)- fuzzy subgroup of G .

DEFINITION 2.5: Let λ and μ be two fuzzy subgroups of G. A proper function F from λ to μ is said to be a homomorphism from λ to μ if for $x,x_1 \in \lambda_{sa}$, $y,y_1 \in \mu_{sa}$

$F(x,y) = \lambda(x)$, $F(x_1,y_1) = \lambda(x_1) \Rightarrow F(xx_1,yy_1) = \lambda(xx_1)$.

A homomorphism F from λ to μ is called an epimorphism if it is (a,b)-surjective .

LEMMA 2.6: Let λ and μ be fuzzy subgroups of G. Let F be a homomorphism from λ to μ . Then

(i) $F(e,e) = \lambda(e)$, where e is the identity element of G .

(ii) If for $x \in \lambda_{sa}$, $y \in \mu_{sa}$,$F(x,y) = \lambda(x)$, then $F(x^{-1},y^{-1}) = \lambda(x^{-1})$.

DEFINITION 2.7: Let λ and μ be two fuzzy subsets of G. The product $\lambda\mu : G \longrightarrow I$ is defined by

$$\lambda\mu(x) = \sup \{ M(\lambda(y),\mu(z)) ; yz = x \} \ \forall\ x \in G .$$

DEFINITION 2.8: Let λ and μ be fuzzy subgroups of G such that $\lambda \le \mu$. Then λ is called a fuzzy subgroup of μ .

3. INTERNAL DIRECT PRODUCT AND SEMIDIRECT PRODUCT:

Let G be a group .

DEFINITION 3.1: A fuzzy subgroup λ of G is said to be a fuzzy normal subgroup of G if

$(x,t) \in_{a} \lambda \Rightarrow (y^{-1}xy,t) \in_{a}\vee q_{(a,b)} \lambda\ \forall\ x,y \in G$ and $\forall\ t \in (a,c]$

or equivalently

$\lambda(y^{-1}xy) \ge M(\lambda(x),k)\ \forall\ y \in G$ and $\forall\ x \in \lambda_{sa}$.

THEOREM 3.2: A fuzzy subset λ of G is a fuzzy normal subgroup of G iff λ_t (λ_{st}) is a normal subgroup of G for all $t \in (a,k]$ ($t \in [a,k)$).

THEOREM 3.3: Let μ, μ_1, μ_2 be fuzzy subgroups of G such that $\mu = \mu_1\mu_2$. Then $\mu_{sa} = (\mu_1)_{sa}(\mu_2)_{sa}$.

DEFINITION 3.4 :Let μ, μ_1, μ_2 be fuzzy subgroups of G . Then μ is said to be the fuzzy internal direct product of μ_1 and μ_2 written as $\mu = \mu_1 \times \mu_2$ if

(i) μ_1 , μ_2 are fuzzy normal .

(ii) $\mu(x) \le a \Longleftrightarrow (\mu_1\mu_2)(x) \le a$

$\mu(x) = (\mu_1\mu_2)(x)$ $\forall$ $x \in \mu_{a,k}$ and $\mu(x) > k \Longleftrightarrow (\mu_1\mu_2)(x) > k$.

(iii) $(\mu_1 \cap \mu_2)_{sa} = \{e\}$.

THEOREM 3.5: Let G be a finite group . A fuzzy subgroup μ of G is a fuzzy internal direct product of two fuzzy subgroups μ_1 , μ_2 iff for all $t \in (a,k]$, μ_t is an internal direct product of $(\mu_1)_t$ and $(\mu_2)_t$.

DEFINITION 3.6:Let G and H be two multiplicative groups and $\alpha : G \longrightarrow Aut(H)$ be a homomorphism of groups mapping $g \in G$ to $\alpha_g \in Aut(H)$. The set $G \times H$ is given the structure of a group by defining $\forall$ h , $h_1 \in H$ and g , $g_1 \in G$,

$$(g,h)(g_1,h_1) = (gg_1 \, , \, \alpha_{g_1}(h)h_1) .$$

This group is called a semidirect product of G and H and is denoted by $G \times_\alpha H$.

THEOREM 3.7: Let λ and μ be two fuzzy subgroups of the groups G and H respectively . Let $\alpha : G \longrightarrow Aut(H)$ be a homomorphism such that μ is $\alpha(G)$-invariant .Then the fuzzy subset $\lambda \times_\alpha \mu$ of $G \times_\alpha H$ given by

$$(\lambda \times_\alpha \mu)(g,h) = M (\lambda(g) , \mu(h))$$

is a fuzzy subgroup of $G \times_\alpha H$, called the fuzzy semidirect product of λ and μ .

4. SUBDIRECT PRODUCT:

Let λ_i be a fuzzy subgroup of a group G_i $\forall$ $i \in I$. $\lambda = \prod \{\lambda_i ; i \in I\} : \prod \{G_i ; i \in I\} \longrightarrow I$ defined by $\lambda[(x_i)] = \inf \{\lambda_i(x_i) ; i \in I\}$ for all $(x_i) \in \prod\{G_i ; i \in I\}$ is a fuzzy subgroup of $\prod \{G_i ; i \in I\}$.

The proper function π_{λ_i} from $\prod \{\lambda_i ; i \in I\}$ to λ_i defined by $\pi_{\lambda_i}((x_i),x) = (\prod\{\lambda_i ; i \in I\})[(x_i)]$ or $\emptyset$ according as $x = x_i$ or $x \neq x_i$ is called the projection mapping from $\prod \{\lambda_i ; i \in I\}$ to λ_i .

Clearly π_{λ_i} is a homomorphism .

DEFINITION 4.1: Let $f : \lambda \longrightarrow \mu$ be a proper function and let λ_o be a fuzzy subset of λ . The proper function $f_o : \lambda_o \longrightarrow \mu$ defined by

$f_o(x,y) = \lambda_o(x)$ or $\emptyset$ according as $f(x,y) = \lambda(x)$ or $\emptyset$, is said to be the restriction of f to λ_o and it is denoted by $f|\lambda_o$.

DEFINITION 4.2: A fuzzy subgroup λ_o of $\prod\{\lambda_i ; i \in I\}$ is called a fuzzy subdirect product of $\prod \{\lambda_i ; i \in I\}$ if $\pi_{\lambda_i}|\lambda_o$ is an epimorphism for all $i \in I$.

THEOREM 4.3: Let λ_i be a fuzzy subgroup of G_i for all $i \in I$
Let λ_o be a fuzzy subgroup of $\prod \{\lambda_i ; i \in I\}$.

(i) If λ_o be a fuzzy subdirect product of $\prod\{\lambda_i ; i \in I\}$, then $(\lambda_o)_{st}$ is a subdirect product of $\prod \{(\lambda_i)_{st} ; i \in I\}$ for all $t \in [a,k)$.

(ii) If $(\lambda_o)_t$ is a subdirect product of $\prod \{(\lambda_i)_t ; i \in I\}$ for all $t \in (a,k]$, then λ_o is a fuzzy subdirect product of $\prod \{\lambda_i ; i \in I\}$.

5. AMALGAMETED PRODUCT:

DEFINITION 5.1: Let G be a group and let λ , μ , γ , δ be fuzzy subgroups of G . δ is said to be a direct product of λ and μ with amalgameted fuzzy subgroup γ denoted by $\delta = \lambda \times_{\gamma} \mu$ if

(i) λ and μ are fuzzy normal,

(ii) $\emptyset \le \delta(x) \le a \iff \emptyset \le (\lambda\mu)(x) \le a$

$\delta(x) = (\lambda\mu)(x)\ \forall\ x \in (\delta)_{a,k}$ and $\delta(x) > k \iff (\lambda\mu)(x) > k$.

(iii) $\emptyset \le (\lambda\cap\mu)(x) \le a \iff \emptyset \le \gamma(x) \le a$

$(\lambda\cap\mu)(x) = \gamma(x)\ \forall\ x \in \gamma_{a,k}$ and $(\lambda\cap\mu)(x) > k \iff \gamma(x) > k$.

(iv) $x \in \lambda_{sa}$, $y \in \mu_{sa} \Rightarrow xy = yx$.

THEOREM 5.2: Let G be a finite group . Let δ , μ , λ , γ be fuzzy subgroups of G such that $\emptyset \le \delta(x) \le a \iff \emptyset \le (\lambda\mu)(x) \le a$ and $\emptyset \le (\lambda\cap\mu)(x) \le a \iff \emptyset \le \gamma(x) \le a$. Then δ is a direct product of λ and μ with amalgameted fuzzy subgroup γ iff δ_t is a direct product of λ_t and μ_t with amalgameted subgroup γ_t for all $t \in (a,k]$.

THEOREM 5.3 : Let G be a finite group . let λ , μ , λ_i , μ_i and γ_i for i = 1,2 be fuzzy subgroups of G . Let λ be a direct product of λ_1 and λ_2 with amalgameted fuzzy subgroup γ_1 and μ be a direct product of μ_1 and μ_2 with amalgameted fuzzy subgroup γ_2. Let there exist an epimorphism f from λ_1 to μ_1 and g from λ_2 to μ_2 satisfying the property (P) : if $x \in (\gamma_1)_{sa} = (\lambda_1)_{sa} \cap (\lambda_2)_{sa}$ and $f(x,y) = \lambda_1(x)$, and $g(x,y_1) = \lambda_2(x)$, where $y, y_1 \in G$, then $y = y_1$.

Then there exists an epimorphism from λ to μ .

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