# Fuzzy Order-Homomorphisms on Groups (II)

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**Abstract**: In this paper, the property of initial L-fuzzy topologies determined by a family of fuzzy order-homomorphisms on groups is investigated. As an application of this result, we prove that the product of a family of L-fuzzy topological groups is a L-fuzzy topological group.

**Keywords**: Fuzzy order-homomorphism on group, initial L-fuzzy topology, L-fuzzy topological groups.

## 1. Introduction

In [2], we introduced the concept of fuzzy order-homomorphism on groups and studied its structures. In this paper, we continue with investigation of fuzzy order-homomorphism on groups. We give a charactrization of the continuity of fuzzy order-homomorphism on groups, and study initial L-fuzzy topological structures determined by a family of fuzzy order-homomorphisms on groups and the product of a family of L-fuzzy topological groups[8].

## 2. Preliminaries

Throughout this paper, $L$, $L_1$, $L_2$ always denote the fuzzy lattices, i.e. completely distributive lattices with order-reversing involutions $\alpha \mapsto \alpha'$. 0 and 1 are their smallest element and greatest element respectively. $L^X$ denotes the family of all L-fuzzy sets[8] on $X$. $\alpha^*_X$ (for short, $\alpha^*$) denotes an $L$-fuzzy set which takes the constsnt value $\alpha \in L$ on $X$. A non-zero element $\lambda$ in $L$ is called a molecule[7] if $\lambda = \alpha \vee \beta$ implies $\lambda = \alpha$ or $\lambda = \beta$, where $\alpha, \beta \in L$. $M(L)$ (or $M$) will denote the set of all molecules in $L$ and $M^*(L^X)$ (or $M^*$) will denote the set of all molecules in $L^X$. $\tilde{X}(L)$ will denote the set of all L-fuzzy points on $X$. We assume that for the empty family $\emptyset$, $\bigvee\emptyset = 0$ and $\bigwedge\emptyset = 1$.

Let $X$ be a group and $A, B \in L^X$. The L-fuzzy sets $A \cdot B$ (simply denoted by $AB$) and $A^{-1}$ on $X$ are defined by

$$(A \cdot B)(x) = \bigvee_{s \cdot t = x} [A(s) \wedge B(t)], \quad A^{-1}(x) = A(x^{-1}),$$

respectively.

**Definition 1**[2]. Let $X$ and $Y$ be two groups. A mapping $F : {L_1}^X \to {L_2}^Y$ is called a fuzzy order-homomorphism on groups if it is an order homomorphism[6] satisfying

$$F(A \cdot B) = F(A) \cdot F(B), \quad \text{for all } A, B \in {L_1}^X$$

In the following, we always assume that $L$ and $L_1$ are regular[6] $X$ and $Y$ are two groups, $e$ denotes the unit element in groups

**Theorem 1**[2]. *The mapping $F : L_1^X \to L_2^Y$ is a fuzzy order-homomorphism on groups if and only if there exist an ordinary group homomorphism $f : X \to Y$ and a finitely meet-preserving order-homomorphism $\varphi : L_1 \to L_2$ such that $F$ is a bi-induced mapping*[4] *of $f$ and $\varphi$, i.e.,*

$$F(A)(y) = \bigvee\{\varphi(A(x)) \mid f(x) = y\}, \quad \forall A \in {L_1}^X, \quad \forall y \in Y. \qquad (2.1)$$

**Corollary 1.** *Let the mapping $f : X \to Y$ be an ordinary group homomorphism. Then the Zadeh's type function*[1] *$f : L^X \to L^Y$ induced by $f$ is a fuzzy order-homomorphism on groups.*

**Remark 1.** Theorem 1.1 show that a fuzzy order-homomorphism on groups $F : {L_1}^X \to {L_2}^Y$ can be defined by an ordinary group homomorphism $f : X \to Y$ and a finitely meet-preserving order homomorphism $\varphi : L_1 \to L_2$. For convenience from now on we usually use $f_\varphi$ instead of $F$.

**Lemma 1.** *Let $f_\varphi : {L_1}^X \to {L_2}^Y$ be a fuzzy order-homomorphism on groups. Then*

(1) $f_\varphi(x_\lambda) = [f(x)]_{\varphi(\lambda)}, \quad \forall x_\lambda \in \tilde{X}(L_1)$.

(2) ${f_\varphi}^{-1}(B)(x) = \varphi^{-1}(B(f(x))), \quad \forall B \in {L_2}^Y, \ \forall x \in X$,

where

$${f_\varphi}^{-1}(B) = \bigvee\{A \in {L_1}^X \mid f_\varphi(A) \le B\} \qquad (2.2)$$

**Lemma 2.** *Let $f_\varphi : {L_1}^X \to {L_2}^Y$ be a fuzzy order-homomorphism on groups. Then*

(1) $f_\varphi^{-1}(f(x)\cdot B)=x\cdot f_\varphi^{-1}(B)$,
$f_\varphi^{-1}(B\cdot f(x))=f_\varphi^{-1}(B)\cdot x,\ \ \forall x\in X,\ \ \forall B\in L_2^Y$.
(2) $f_\varphi^{-1}(A)\cdot f_\varphi^{-1}(B)\subset f_\varphi^{-1}(AB),\ \ \forall A,\ B\in L_2^Y$

**Definition 2.** A L-fuzzy topology on a set $X$ is a family $\delta$ of L-fuzzy subsets of $X$ which satisfies the following conditions:
(1) $\alpha^*\in\delta$ for all $\alpha\in L$;
(2) $\delta$ is closed under arbitrary unions;
(3) $\delta$ is closed under finit intersections.

The pair $(L^X,\delta)$ (or $(X,\delta)$, for simplicity) is called L-fuzzy topological space and the members of $\delta$ are called open L-fuzzy sets. When $A\in\delta, A'$ is called a closed L-fuzzy set.

For the notions of R-neighborhood, R-neighborhood base of a molecule $x_\lambda$ in $(L^X,\delta)$ and continuous order homomorphism can be found in [7]. Let $(L^X,\delta_1)$ and $(L^Y,\delta_2)$ be two L-fuzzy topological spaces. The mapping $f:X\to Y$ is said to be continuous, if the Zadeh's type function[5] of it is a continuous order homomorphism from $(L^X,\delta_1)$ into $(L^Y,\delta_2)$.

**Definition 3**[8]. Let $X$ be a group and $\delta$ be a L-fuzzy topology on $X$. The pair $(L^X,\delta)$ (or $(X,\delta)$) is said to be a L-fuzzy topological group iff the following conditions are satisfied:
(a) The mapping $g:(X,\delta)\times(X,\delta)\to(X,\delta),\ \ (x,y)\mapsto xy$ is continuous;
(b) The mapping $h:(X,\delta)\to(X,\delta),\ \ x\mapsto x^{-1}$ is continuous.

Lemma 3.3 in [8] given some of the propertis of R-neighborhood base of $e_\lambda$ in L-fuzzy topological group. It must be pointed out that the conclusion (6) of Lemma 3.3 should be modified as "for all $\mu\in L$, if $\lambda\not\leq\mu$, then there exists a $V\in\eta_\lambda$ such that $\mu^*\leq V$". By Lemma 3.3 and Theorem 3.1 in [8], we obtain the following:

**Theorem 2** *Let $(L^X,\delta)$ be a L-fuzzy topological group. If $\eta_\lambda=\{U\}$ is a R-neighborhood base of $e_\lambda$ for each $\lambda\in M(L)$, then we have*
(1) *if $U\in\eta_\lambda$, then $e_\lambda\notin U$;*
(2) *if $U,V\in\eta_\lambda$, then there exists a $W\in\eta_\lambda$ such that $U\cup V\subset W$;*
(3) *for each $U\in\eta_\lambda$, there exists a $V\in\eta_\lambda$ such that $V'\cdot V'\subset U'$;*
(4) *for each $U\in\eta_\lambda$, there exists a $V\in\eta_\lambda$ such that $U\subset V^{-1}$;*
(5) *for each $U\in\eta_\lambda$ and each $x\in X$, there exists a $V\in\eta_\lambda$ such that $x^{-1}V'x\subset U'$;*
(6) *for such $U\in\eta_\lambda$, if $x_\alpha\notin U(\alpha\in M)$, then there exists a $V\in\eta_\alpha$ such that $xV'\subset U'$;*

*(7) for all $\mu \in L$, if $\lambda \not\leq \mu$ then there exists a $V \in \eta_\lambda$ such that $\mu^* \leq V$.*

*Conversely, let $X$ be a group, if for each $\lambda \in M(L)$ there is a family $\eta_\lambda = \{U\}$ of L-fuzzy sets on $X$ which satisfies the above conditions (1) – (7), then there is a L-fuzzy topology $\delta$ on $X$ such that $(L^X, \delta)$ is a L-fuzzy topological group and the $\eta_\lambda$ is a R-neighborhood base of $e_\lambda$ in $(L^X, \delta)$.*

## 3. Main results

**Theorem 3.** *Let $(L_1^X, \delta_1)$ and $(L_2^Y, \delta_2)$ be two L-fuzzy topological groups, and let $f_\varphi : L_1^X \to L_2^Y$ be a fuzzy order-homomorphism on groups. Then $f_\varphi$ is continuous iff $f_\varphi$ is continuous at $e_\lambda$ for each $\lambda \in M(L)$.*

**Proof.** The necessity is evident (See [7, Theorem 2.6.3]).

Sufficiency. By [7, Theorem 2.6.3], it is enough to show that for each $x_\lambda \in M^*(L_1^X)$, $f_\varphi$ is continuous at $x_\lambda$.

Let $A$ be a R-neighborhood of $[f(x)]_{\varphi(\lambda)}$. Since $(L_2^Y, \delta_2)$ be a L-fuzzy topological group, by [8, Lemma 3.1], there exists a $P \in \eta^{(2)}_{\varphi(\lambda)}$ (where $\eta^{(2)}_{\varphi(\lambda)}$ is a R-neighborhood base of $e_{\varphi(\lambda)}$ in $(L_2^Y, \delta_2)$) such that $A \subset f(x)P$. By the continuity of $f_\varphi$ at $e_\lambda$, we know that $(f_\varphi)^{-1}(P)$ is a R-neighborhood of $e_\lambda$. From Lemma 2 it follows that

$$f_\varphi{}^{-1}(A) \subset f_\varphi{}^{-1}(f(x)P) = x f_\varphi{}^{-1}(P).$$

Thus $f_\varphi{}^{-1}(A)$ is a R-neighborhood of $x_\lambda$. Therefore by [7, Definition 2.6.2] $f_\varphi$ is continuous at $x_\lambda$.

**Theorem 4.** *Let $X$ be a group, $\{(L_2^{X_\alpha}, \delta_\alpha) \mid \alpha \in \Gamma\}$ a family of L-fuzzy topological group, and let the mapping $(f_\alpha)_{\varphi_\alpha} : L_1^X \to L_2^{X_\alpha}$ be a fuzzy order-homomorphism on groups for each $\alpha \in \Gamma$. By $\delta$ we denote the weakest L-fuzzy topology on $X$ with respect to which each of the mappings $(f_\alpha)_{\varphi_\alpha} (\alpha \in \Gamma)$ is continuous. If each $\varphi_\alpha$ $(\alpha \in \Gamma)$ is a injection, then $(L_1^X, \delta)$ is a L-fuzzy topological group. $\delta$ is called the L-initial fuzzy topology determined by $\{(f_\alpha)_{\varphi_\alpha} \mid \alpha \in \Gamma\}$.*

**Proof.** For each $\lambda \in M(L_1)$, we define a family $\eta_\lambda$ of L-fuzzy sets on $X$ as follows:

$$\eta_\lambda = \{\bigcup_{i=1}^{n} (f_{\alpha_i})^{-1}_{\varphi_{\alpha_i}}(P_{\alpha_i}) \mid P_{\alpha_i} \in \tilde{\eta}_{\varphi_{\alpha_i}(\lambda)},\ i = 1, 2, \ldots, n;\ n \in N\}, \tag{3.1}$$

where $\tilde{\eta}_{\varphi_{\alpha_i}(\lambda)}$ is a R-neighborhood base of $e_{\varphi_{\alpha_i}(\lambda)}$ in $(L_2^{X_{\alpha_i}}, \delta_{\alpha_i})$. Using Lemma 1 and Lemma 2, we can show that $\eta_\lambda$ satisfies the conditions (1) – (7) in Theorem 2. Thus, by Theorem 2 there is a L-fuzzy topology $\delta$ on $X$ such that $(L_1^X, \delta)$ is a L-fuzzy topological group, and $\eta_\lambda$ is a R-neighborhood base of $e_\lambda$ in $(L_1^X, \delta)$. From the definition of $\eta_\lambda$ (See (3.1)), it is easy to know that $\delta$ is the weakest L-fuzzy topology on $X$ with respect to which each of the mappings $(f_\alpha)_{\varphi_\alpha}$ ($\alpha \in \Gamma$) is continuous.

**Corollary 2.** *Let $\{(L^{X_t}, \delta_t)\}_{t\in T}$ be a family of L-fuzzy topological groups, and let $X = \prod_{t\in T} X_t$ be the direct product of the algebraic groups $X_t$ ($t \in T$) and $\delta$ be the product of L-fuzzy topologies $\{\delta_t\}_{t\in T}$. Then $(L^X, \delta)$ is a L-fuzzy topological group, and it is called the product of L-fuzzy topological groups $(L^{X_t}, \delta_t)$ ($t \in T$).*

**Proof.** Let $\delta$ be the weakest L-fuzzy topology on $X$ for which each projection $\tilde{P}_t : L^X \to L^{X_t}$ is continuous, where $\tilde{P}_t$ is the Zadeh's type function induced by the projective mapping $P_t : X \to X_t$. By [7, Theorem 2.8.10], we know that $\delta$ is the product of L-fuzzy topologies $\{\delta_t\}_{t\in T}$. Since the projection $P_t : X \to X_t$ is an ordinary group homomorphism, by Corollary 1 the Zadeh's type function $\tilde{P}_t : L^X \to L^{X_t}$ is a fuzzy order homomorphism on groups. Hence from Theorem 4 $(L^X, \delta)$ is a L-fuzzy topological group.

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