SOME RESULTS ON UNION OF FUZZY SUBGROUPS

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Abstract

This paper has answered the question whether a true fuzzy subgroup can be represented union of two mutually unequivalent true fuzzy subgroups proposed by [1]. Some results are obtained.

Keywords: Fuzzy subgroups; union; true fuzzy subgroups

1. Instruction

It is well-known in the ordinary group theory that any group can not be represented union of two true subgroups. The fact was generalized to fuzzy group by Rosenfeld[2], that is, any group can not be represented union of two true fuzzy subgroups where the group means characteristic function of a group G. Dixit, KUMAR and AJMAL[1] gave their further study for this problem. The question whether a true fuzzy subgroup θ of G can be represented union of two mutually exclusive true fuzzy subgroups was put forward and given a complete answer. They stated that the answer depended on the image set $I_m\theta$ of θ where $I_m\theta=\{\theta(x) \mid x\in G\}$. When $I_m\theta$ includes at least two nonzero elements, θ can be always represented union of two mutually exclusive true fuzzy subgroups. This conclusion is not identical with correspondent one in ordinary group theory. But when $|I_m\theta|=1$ or $I_m\theta=\{0,t\}$, $0<t\leqslant 1$, we can obtain identical conclusion with ordinary group theory, that is, θ can not be represented union of two mutually exclusive true fuzzy subgroups.

This paper will answer whether a true fuzzy subgroup can be represented union of two mutually unequivalent true fuzzy subgroups which is proposed by Dixit et al[1](in the need of mutually exclusive true fuzzy subgroups certainly——authors of this paper).

2. Preliminaries

Definition 1. Let θ be a fuzzy subgroup of a group G. If $|I_m \theta| \neq 1$(that is, θ is not constant), then θ is called a true fuzzy subgroup, otherwise a untrue fuzzy subgroup.

Definition 2. Two fuzzy subgroups μ and η of a group G are called equivalent if their level groups are completely the same, that is

$$\{ \mu_\lambda \mid \lambda \in [0,1] \} = \{ \eta_\lambda \mid \lambda \in [0,1] \} .$$

Otherwise, μ and η are not equivalent(or unequivalent).

Lemma 1. Let θ be a fuzzy subset of a group G. Then θ is a fuzzy subgroup if and only if every level subset θ_λ of θ is a subgroup of G.

Lemma 2. Let θ be a fuzzy subgroup of G and $t_1, t_2 \in I_m \theta, t_1 > t_2$. Then fuzzy subset μ of G defined as follows

$$\mu(x) = \begin{cases} t_2 & t_1 > \theta(x) \geqslant t_2 \\ \theta(x) & \text{otherwise} \end{cases}$$

is also a fuzzy subgroup of G.

Proof: Take $\lambda \in [0,1]$. When $\lambda > t_1$ or $\lambda \leqslant t_2$, $\mu_\lambda = \theta_\lambda$ obviously. When $t_1 > \lambda > t_2$, since

$$x \in \mu_\lambda \iff \mu(x) \geqslant \lambda > t_2 \iff \mu(x) = \theta(x) \geqslant t_1 \iff x \in \theta_{t_1}$$

therefore $\mu_\lambda = \theta_{t_1}$

In a word, every level cut set of μ is a certain level cut set of θ. Since θ is a fuzzy subgroup, from Lemma 1, μ is also a fuzzy subgroup.

3. Union of fuzzy subgroup

We discuss above question in several conditions.

(1) When $|I_m \theta| = 1$ (Dixit calls θ not a true fuzzy subgroup) or $I_m \theta = \{0, t\}$, $t>0$, from [1], we know that θ can not be decomposed union of two mutually exclusive true fuzzy subgroups. Hence, certainly, θ can not be decomposed union of two unequivalent true subgroups.

(2) when $I_m \theta = \{t_1, t_2\}, t_1>t_2>0$, θ have only two level subgroups θ_{t_1}, θ_{t_2} $(=G)$ and $\theta_{t_1} \subseteq \theta_{t_2}$.

If θ_{t_1} has true subgroup H, we define fuzzy subgroups μ and η as follows

$$\mu(x) = \begin{cases} t_1 & x \in H \\ t_2 & x \in H \end{cases} \qquad \eta(x) = \begin{cases} t_1 & x \in \theta_{t_1} \\ t_2 - \varepsilon & \theta(x) = t_2 \end{cases}$$

where $0 < \varepsilon < t_2$. From Lemma 2, μ and η are fuzzy subgrouups of G and $\theta = \mu \vee \eta$ where μ and η are neither equivalent nor mutually inclusive.

If G has true subgroup K such that $\theta_{t_1} \subseteq K \subseteq G$, then μ and η defined as follows

$$\mu(x) = \begin{cases} t_1 & x \in \theta_{t_1} \\ t_2 & x \in K \text{ but } x \notin \theta_t \\ t_2 - \varepsilon & x \in \theta_{t_2} \text{ but } x \notin K \end{cases} \qquad \eta(x) = \begin{cases} t_1 - \delta & \theta(x) = t_1 \\ t_2 & \theta(x) = t_2 \end{cases}$$

are two neither equivalent nor mutually inclusive true fuzzy subgroups and $\theta = \mu \vee \eta$.

Conversely, if θ can be decomposed union of two mutually unequivalent true fuzzy subgroups, then there exists at least a level subgroup denoted by μ_* among level subgroups of μ and η which is not θ_{t_1} and θ_{t_2}

If $\mu_* \cap \theta_{t_1} = \theta_{t_2}$, then $\theta_{t_1} \subseteq \mu_*$, that is, G has subgroup $K = \mu_*$ such that $\theta_{t_1} \subseteq K \subseteq \theta_{t_2}$. If $\theta_{t_1} \cap \mu_* \subseteq \theta_{t_1}$ then $H = \theta_{t_1} \cap \mu_*$ is a true subgroup of θ_{t_1}.

Summing up, we have following theorem.

Theorem 1. Let θ be a true fuzzy subgroup of a group G and $I_m \theta = \{t_1, t_2\}$, $t_1>t_2>0$. Then θ can be represented union of two mutually unequivalent true fuzzy subgroups if and only if G has true subgroup H such that $H \subseteq \theta_*$ or $\theta_* \subseteq H$.

(3) $| I_m \theta | \geqslant 3$.

1° $I_m \theta = \{t_1, t_2, t_3\}$, $t_1 > t_2 > t_3$. Let

$$\mu(x) = \begin{cases} t_2 & \theta(x) \geqslant t_2 \\ t_3 & \theta(x) = t_3 \end{cases} \qquad \eta(x) = \begin{cases} t_1 & \theta(x) = t_1 \\ t_3 & t_1 > \theta(x) \geqslant t_3 \end{cases}$$

From Lemma 2, we know that μ and η are true fuzzy subgroups which are neither mutually inclusive nor equivalent obviously and $\theta = \mu \vee \eta$.

2° $| I_m \theta | \geqslant 4$ (which include $| I_m \theta | = +\infty$). Take t_1, t_2, t_3 and $t_4 \in I_m \theta$ such that $t_1 > t_2 > t_3 > t_4$. Let

$$\mu(x) = \begin{cases} t_3 & t_1 > \theta(x) \geqslant t_3 \\ \theta(x) & \text{otherwise} \end{cases} \qquad \eta(x) = \begin{cases} t_4 & t_2 > \theta(x) \geqslant t_4 \\ \theta(x) & \text{otherwise} \end{cases}$$

From Lemma 2, μ and η are true fuzzy subgroups which are neither mutually inclusive nor equivalent and $\theta = \mu \vee \eta$. Therefore, we obtain following theorem.

Theorem 2. Let θ be true fuzzy subgroup of a group G. If $| I_m \theta | \geqslant 3$, then θ can be represented union of two mutually unequivalent true fuzzy subgroups.

References

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