CORRESPONDENCE RELATION OF FUZZY SUBGROUPS IN TWO HOMOMORPHISM GROUPS

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Abstract
In this paper, we prove the correspondence theorem for fuzzy subgroups in homomorphism groups and derive the structure of fuzzy quotient groups.

Keywords: Fuzzy subgroup, Fuzzy invariant subgroup, Fuzzy quotient group

We make appointment as follows: The notations $F(X)$, $GF(X)$ and $NGF(X)$ are sets of all fuzzy subsets, fuzzy subgroups and fuzzy invariant subgroups respectively. The definitions[1] of fuzzy quotient group and fuzzy quotient subgroup were given as follows by Luo in 1986.
Definition 1 Suppose that $\underset{\sim}{N} \in NGF(X)$. Let

$$X/\underset{\sim}{N} = \{ a\underset{\sim}{N} \mid a \in X\}.$$

We stipulates that $a\underset{\sim}{N} \circ b\underset{\sim}{N} = (ab)\underset{\sim}{N}$. Then $(X/\underset{\sim}{N}, \circ)$ is a group. We call it fuzzy quotient group of X with respect to $\underset{\sim}{N}$ (unit element is $e\underset{\sim}{N}=\underset{\sim}{N}$, $(a\underset{\sim}{N})^{-1}=a^{-1}\underset{\sim}{N}$).
Definition 2. Suppose that P: $X \to X/\underset{\sim}{N}$ is natural homomorphism and $\underset{\sim}{H}$ is fuzzy subgroup of X. Then $P(\underset{\sim}{H})$ is called fuzzy queotient subgroup of $\underset{\sim}{H}$ with respect to $\underset{\sim}{N}$ denoted by $\underset{\sim}{H}/\underset{\sim}{N}$
It is well-known that in ordinary group theory[3] we have following conclusions: (1) Suppose that f: $X \to X'$ be epimorphism. Then subgroups of X which include $Kerf$ and all subgroups of X' are correspondent.

(2) If $N \trianglelefteq X$, then there exists only subgroup H of X such that $N \subseteq H$ and $H' = H/N$ for every subgroup H' of X/N. Especially, when $H' \trianglelefteq X/N$, $H \trianglelefteq X$ and $X/N \diagup H/N \cong X/N$. In this paper, we generalize above conclusions to fuzzy group theory.

Theorem 1. Suppose that $f: X \to X'$ be epimorphism map, M set of fuzzy subgroups of X whose membership degrees on Kerf are constant and M' set of all fuzzy subgroups of X', that is,

$$M = \{\underset{\sim}{H} \mid \underset{\sim}{H} \in GF(X) \text{ and } \underset{\sim}{H}(x) = a \in [0,1],\ \forall\ x \in Kerf\},\ M' = GF(X').$$

Then $\psi: \underset{\sim}{H} \to f(\underset{\sim}{H})$ is order-preserving bijective map of M to M'. When $\underset{\sim}{N} \in M$ is fuzzy invariant subgrup of X, $\underset{\sim}{N}' = f(\underset{\sim}{N})$ is fuzzy invariant subgroup of X' and

$$X/\underset{\sim}{N} \cong X'/\underset{\sim}{N}'.$$

To prove Theorem 1, we first prove the following Lemma.

Lemma. Suppose that K be invariant subgroup of group X and $\underset{\sim}{H}$ fuzzy subgroup of X. If the membership degree of $\underset{\sim}{H}$ on K is constant and maximal, then the membership degree of $\underset{\sim}{H}$ on every coset aK of K is all constant $\underset{\sim}{H}(a)$.

Proof: $\forall\ k \in K$, $\underset{\sim}{H}(ak) \geqslant \min\{\underset{\sim}{H}(a), \underset{\sim}{H}(k)\} = \underset{\sim}{H}(a)$,

$\underset{\sim}{H}(a) = \underset{\sim}{H}(ak \cdot k^{-1}) \geqslant \min\{\underset{\sim}{H}(ak), \underset{\sim}{H}(k^{-1})\} = \min\{\underset{\sim}{H}(ak), \underset{\sim}{H}(k)\} = \underset{\sim}{H}(ak)$

therefore $\quad \underset{\sim}{H}(ak) = \underset{\sim}{H}(a)$

We prove Theorem 1 as follows:

Proof: From Theorem 7 of [1], we know that ψ is a map of M to M'. For $\forall\ \underset{\sim}{H}' \in M' = GF(X')$, let $\underset{\sim}{H} = f^{-1}(\underset{\sim}{H}')$. From Theorem 8 of [1], we know that $\underset{\sim}{H} \in GF(X)$ and for $\forall\ a \in Kerf$

$$\underset{\sim}{H}(a) = f^{-1}(\underset{\sim}{H}')(a) = \underset{\sim}{H}'(f(a)) = \underset{\sim}{H}'(e') \text{ (constant)}$$

Therefore $\underset{\sim}{H} \in M$. It is easy to know that

$$\psi(\underset{\sim}{H}) = f(\underset{\sim}{H}) = f(f^{-1}(\underset{\sim}{H}')) = \underset{\sim}{H}'.$$

ψ is a epimorphism of M to M'.

Take $\underset{\sim}{H}, \underset{\sim}{K} \in M$. If $\psi(\underset{\sim}{H}) = \psi(\underset{\sim}{K})$ then for $\forall\ x \in X$ $\psi(\underset{\sim}{H})(x') = \psi(\underset{\sim}{K})(x')$ where $x' = f(x)$, that is, $f(\underset{\sim}{H})(x') = f(\underset{\sim}{K})(x')$,

$$\sup_{f(y)=f(x)=x'} \underset{\sim}{H}(y) = \sup_{f(y)=f(x)=x'} \underset{\sim}{K}(y), \qquad \sup_{y \in xK} \underset{\sim}{H}(y) = \sup_{y \in xK} \underset{\sim}{K}(y).$$

Since membership degrees of $\underset{\sim}{H},\underset{\sim}{K}$ *on* $K=Kerf$ *are constant. they are* $\underset{\sim}{H}(e)$. $\underset{\sim}{K}(e)$ *and maximal. From Lemma, above equality becomes*

$$\underset{\sim}{H}(x)=\underset{\sim}{K}(x)$$

that is, $\underset{\sim}{H}=\underset{\sim}{K}$. ψ *is monomorphism.*

If $\underset{\sim}{H},\underset{\sim}{K}\in M$, $\underset{\sim}{H}\subseteq\underset{\sim}{K}$, *then for* $\forall\ x'\in X'$,

$$\psi(\underset{\sim}{H})(x')=f(\underset{\sim}{H})(x')=\sup_{f(x)=x'}\underset{\sim}{H}(x)\leqslant\sup_{f(x)=x'}\underset{\sim}{K}(x)$$

$$=f(\underset{\sim}{K})(x')=\psi(\underset{\sim}{K})(x')$$

Therefore $\psi(\underset{\sim}{H})\subseteq\psi(\underset{\sim}{K})$, ψ *is order-preserving.*

Summing up, ψ *is order-preserving bijective map.*

The other part of Theorem 1 can be obtained from Theorem 7 and Theorem 8 of [1] directly.

The following Theorem give the structure of fuzzy subgroup of $X/\underset{\sim}{N}$.

Theorem 2. Suppose that $\underset{\sim}{N}$ *be fuzzy invariant subgroup of group* X *and* $X/\underset{\sim}{N}$ *fuzzy quotient group of* X *with respect to* $\underset{\sim}{N}$. *Then for every fuzzy subgroup* $\underset{\sim}{H}'$ *of* $X/\underset{\sim}{N}$, *there exists only fuzzy subgroup* $\underset{\sim}{H}$ *such that the membership degree of* $\underset{\sim}{H}$ *on* $Ne=\{x\mid x\in X,\underset{\sim}{N}(x)=\underset{\sim}{N}(e)\}$ *is constant*$(\underset{\sim}{H}(e))$ *and* $\underset{\sim}{H}/\underset{\sim}{N}=\underset{\sim}{H}'$. *Especially, when* $\underset{\sim}{H}'$ *is fuzzy invariant subgroup of* $X/\underset{\sim}{N}$, H *is also fuzzy invariant subgroup of* X, *and*

$$X/\underset{\sim}{N}\cong X/\underset{\sim}{N}\,/\,\underset{\sim}{H}/\underset{\sim}{N}.$$

Proof: Suppose that P: $X\to X/\underset{\sim}{N}$ *be natural homemorphism. Then*

$$K=Kerp=\{x\in X\mid P(x)=\underset{\sim}{N}\}=\{x\in X\mid x\underset{\sim}{N}=\underset{\sim}{N}\}=\{x\in X\mid x\in Ne\}=Ne.$$

Let $\underset{\sim}{H}=P^{-1}(\underset{\sim}{H}')$. *Since* $\underset{\sim}{H}'\in GF(X/\underset{\sim}{N}')$, *from Theorem 1,* $\underset{\sim}{H}$ *is fuzzy subgroup of* X *whose membership degree on* Ne *is constant and*

$$\underset{\sim}{H}/\underset{\sim}{N}=P(\underset{\sim}{H})=P(P^{-1}(\underset{\sim}{H}'))=\underset{\sim}{H}'.$$

From Theorem 1, we know that $\underset{\sim}{H}$ *is only determined by* $\underset{\sim}{H}'$. *When* $\underset{\sim}{H}'\in NGF(X/\underset{\sim}{N})$, $\underset{\sim}{H}\in NGF(X)$ *and*

$$X/\underset{\sim}{H}\cong X/\underset{\sim}{H}'\,/\,\underset{\sim}{H}'=X/\underset{\sim}{N}\,/\,\underset{\sim}{H}/\underset{\sim}{N}$$

If fuzzy invariant subgroup $\underset{\sim}{N}(=N)$ *is ordinary invariant subgroup, then* $Ne=N$. *Therefore, we have*

Corollary. Suppose that N *be a ordinary invariant subgroup of* X. *Then for every fuzzy invariant subgroup* $\underset{\sim}{H}'$ *of* $X/\underset{\sim}{N}$, *there exists only fuzzy subgroup* $\underset{\sim}{H}$ *of* X *such that the membership degree of* $\underset{\sim}{H}$ *on* N *is* *constant and* $\underset{\sim}{H}/N=\underset{\sim}{H}'$.

In ordinary group theory, only when subgroup H of X include invariant subgroup N do we consider quotient group of H with respect to N. For arbitrary fuzzy subgroup $\underset{\sim}{H}_1$ *of X Definition 2 define fuzzy quotient subgroup* $\underset{\sim}{H}_1/\underset{\sim}{N}$. *From Theorem 2 we know that* $\underset{\sim}{H}_1/\underset{\sim}{N}$ *must equal certain* $\underset{\sim}{H}/\underset{\sim}{N}$ *where membership degree of* $\underset{\sim}{H}$ *on Ne is constant. Therefore we may only define fuzzy quotient* $\underset{\sim}{H}/\underset{\sim}{N}$ *for fuzzy subgroup* $\underset{\sim}{H}$ *whose membership degree on Ne is constant. This makes* $\underset{\sim}{H}/\underset{\sim}{N}$ *and* $\underset{\sim}{H}$ *determined mutually and identical to ordinary group theory.*

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