FUZZY TOPOLOGY ON FUZZY SETS: PRODUCT FUZZY TOPOLOGY AND FUZZY TOPOLOGICAL GROUPS

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1. INTRODUCTION:

The notion of fuzzy topology on fuzzy sets was introduced by Chakrabarty and Ahsanullah [1]. Chaudhury and Das [3] studied several fundamental properties of such fuzzy topologies. However the concept of product fuzzy topology has not yet been introduced in the new setting. The object of the present paper is to precisely do that and examine the product - invariance of some fundamental properties. Of course, here the condition $\emptyset, \lambda \in \tau$ (λ is a fuzzy subset of a set X) in [1] has been replaced by the condition $k \cap \lambda \in \tau$ for all $k \in [0,1]$. The reasons for this change are the same as those Lowen [5] gave for the replacement of the condition $\emptyset, 1 \in \tau$ in Chang's [2] fuzzy topology by the condition $k \in \tau$ for all $k \in [0,1]$. It is proved that if (λ,τ) and (μ,σ) be fuzzy topological spaces, then (i) fuzzy Hausdorffness of (λ,τ) and (μ,σ) implies fuzzy Hausdorffness of $(\lambda\times\mu,\tau\times\sigma)$ and the reverse implication is true if an additional condition is satisfied, (ii) fuzzy compactness of $(\lambda\times\mu,\tau\times\sigma)$ implies the fuzzy compactness of (λ,τ) and (μ,σ) (under certain conditions) but the reverse implication is not true and (iii) in the case of fuzzy connectedness neither implication is true. The concept of product fuzzy topology is then used to define fuzzy group topology on a fuzzy subgroup of a group G. Some fundamental properties of fuzzy topological groups are obtained in the new setting.

2. PRELIMINARIES:

Let X be a non-empty set and let $I = [0,1]$. I^X will denote the set of all functions $\alpha: X \longrightarrow I$. A member of I^X is called a fuzzy subset of X.

If $k \in I$, k_X or simply k will denote the fuzzy subset of X taking the value k all over X. If $X = \{x_1,\ldots,x_n,\ldots\}$ and if $\lambda \in I^X$ be such that $\lambda(x_i) = a_i$ for $i = 1,\ldots,n\ldots$, then λ will be denoted by $(a_1,\ldots,a_n,\ldots)$.

If $x \in X$ and $p \in (0,1]$, by the fuzzy point x_p we mean the fuzzy subset of X which takes the value p at the point x and 0 elsewhere.

If $\lambda \in I^X$ and $\lambda(x) \geq p$ (or $> p$), then we write $x_p \in$ (or $\in_1$) λ.

$M(x,y)$ will denote the minimum of x and y.

DEFINITION 2.1:[1] Let λ be a fuzzy subset of X. A collection τ of fuzzy subsets of λ satisfying

(i) $k \cap \lambda \in \tau \ \forall \ k \in I$,

(ii) $\mu_i \in \tau \ \forall \ i \in \Delta \Longrightarrow \cup\{\mu_i;\ i \in \Delta\} \in \tau$,

(iii) $\mu,\nu \in \tau \Longrightarrow \mu \cap \nu \in \tau$

is called a fuzzy topology on λ. The pair (λ,τ) is called a fuzzy topological space. Members of τ will be called fuzzy open sets and their complements w.r.to λ are called fuzzy closed sets of (λ,τ).

REMARK 2.2: Here the condition (i) in Chakrabarty and

Ahsanullah's [1] definition viz. $\emptyset, \lambda \in \tau$ has been changed. The reasons for this change are the same as those given by Lowen [5] for replacing the condition $\emptyset, 1 \in \tau$ in Chang's [2] fuzzy topology by the condition $k \in \tau$ for all $k \in I$.

Unless otherwise mentioned by a fuzzy topological space we shall mean it in the sense of Definition 2.1 and (λ, τ) will denote a fuzzy topological space.

If $\mathcal{B}$ be a given collection of fuzzy subsets of λ, then the family of all possible unions and finite intersections of the members of $\mathcal{B}$ and the family $\{\lambda \cap k;\ k \in I\}$ is a fuzzy topology on λ and it will be denoted by $\tau(\mathcal{B})$.

DEFINITION 2.3: $\mathcal{B} \subset \tau$ is called an open base of τ if every member of τ can be expressed as the union of some members of $\mathcal{B}$.

DEFINITION 2.4:[3] (λ, τ) is said to be a fuzzy Hausdorff space if $\forall\ x_p,\ y_q \in_1 \lambda$ $(x \neq y)$, $\exists\ \mu, \nu \in \tau$ s.t. $x_p \in_1 \mu$, $y_q \in_1 \nu$ and $\mu \cap \nu = \emptyset$.

DEFINITION 2.5: (λ, τ) is said to be fuzzy compact if $\forall\ \mathcal{B} \subset \tau$ satisfying $\cup\{\mu;\ \mu \in \mathcal{B}\} = \lambda$ and $\forall\ \varepsilon > \emptyset$, $\exists$ a finite subcollection $\mathcal{B}_0$ of $\mathcal{B}$ such that $\cup\{\mu;\ \mu \in \mathcal{B}_0\} \geq \lambda_\varepsilon$ where λ_ε is defined by $\lambda_\varepsilon(x) = \lambda(x) - \varepsilon$ or $\emptyset$ according as $\lambda(x) > \varepsilon$ or $\lambda(x) \leq \varepsilon$.

DEFINITION 2.6: A fuzzy subset μ of λ is said to be fuzzy separated if $\exists\ \nu, \delta \in \tau$ such that $\mu = \nu \cup \delta$, $\mu \neq \nu$, $\mu \neq \delta$, $\nu \cap \delta = \emptyset$.

DEFINITION 2 7: (λ, τ) is said to be fuzzy connected if no fuzzy clopen subset of (λ, τ) can be fuzzy separated.

Let X and Y be two non-empty sets and let $\lambda \in I^X$, $\mu \in I^Y$

DEFINITION 2.8:[1] A fuzzy subset F of $X \times Y$ is said to be a fuzzy proper function from λ to μ if

(i) $F(x,y) \leq M(\lambda(x),\mu(y)) \ \forall \ (x,y) \in X \times Y$,

(ii) $\forall \ x \in X, \ \exists \ y_0 \in Y$ such that $F(x,y_0) = \lambda(x)$ and $F(x,y) = \emptyset$ if $y \neq y_0$.

Let $F: \lambda \longrightarrow \mu$ be a proper function. If $A \leq \lambda$, $B \leq \mu$, then $F(A)$, $F^{-1}(B)$ are defined by

$(F(A))(y) = \sup.\{M(F(x,y),A(x)); \ x \in X\} \ \forall \ y \in Y$,

$(F^{-1}(B))(x) = \sup.\{M(F(x,y),B(y)); \ y \in Y\} \ \forall \ x \in X$.

DEFINITION 2.9: A proper function $F: \lambda \longrightarrow \mu$ is said to be

(i) injective if $F(x_1,y) = \lambda(x_1)(\neq \emptyset)$, $F(x_2,y) = \lambda(x_2)(\neq \emptyset) \Longrightarrow x_1 = x_2 \ \forall \ x_1,x_2 \in X, \ y \in Y$;

(ii) surjective if $\forall \ y \in Y$ with $\mu(y) \neq \emptyset$, $\exists \ x \in X$ such th $F(x,y) = \lambda(x)$,

(iii) bijective if F is both injective and surjective.

If F be injective, then $\forall \ \delta \leq \lambda$, $F^{-1}(F(\delta)) = \delta$.

DEFINITION 2.10: The proper function $I_\lambda: \lambda \longrightarrow \lambda$ defined by $I_\lambda(x,y) = \lambda(x)$ or $\emptyset$ according as $y = x$ or $y \neq x$ is said to be the identity proper function on λ.

DEFINITION 2.11: If $F: \lambda \longrightarrow \mu$ and $G: \mu \longrightarrow \nu$ ($\nu \in I^Z$) be proper functions, then the proper function $GF: \lambda \longrightarrow \nu$ is defined by

$$(GF)(x,z) = \begin{cases} \lambda(x) \text{ if } \exists \ y \in Y \text{ such that } F(x,y) = \lambda(x) \\ \qquad G(y,z) = \mu(y), \\ \emptyset, \text{ otherwise.} \end{cases}$$

DEFINITION 2.12: $G: \mu \longrightarrow \lambda$ is called an inverse of Φ: $F:\lambda \longrightarrow \mu$ if $GF = I_\lambda$, $FG = I_\mu$.

DEFINITION 2.13: A proper function $F: (\lambda,\tau) \longrightarrow (\mu,\tau_1)$ is said to be

(i) fuzzy continuous if $F^{-1}(\nu) \in \tau\ \forall\ \nu \in \tau_1$,

(ii) fuzzy open if $F(\delta) \in \tau_1\ \forall\ \delta \in \tau$,

(iii) fuzzy homeomorphism if F be bijective, fuzzy continuous and open.

THEOREM 2.14: Let $F: (\lambda,\tau) \longrightarrow (\mu,\tau_1)$ be fuzzy continuous.

(i) If $\gamma \le \lambda$ be fuzzy compact in (λ,τ), then $F(\gamma)$ is fuzzy compact in (μ,τ_1);

(ii) If F be invertible, then the fuzzy connectedness of γ in (λ,τ) implies the fuzzy connectedness of $F(\gamma)$ in (μ,τ_1).

DEFINITION 2.15: Let G be a group. A fuzzy subset λ of G is said to be a fuzzy subgroup of G if $\forall\ x,y \in G$

(i) $\lambda(xy) \ge M(\lambda(x),\lambda(y))$,

(ii) $\lambda(x^{-1}) = \lambda(x)$.

DEFINITION 2.16: An element $a \in X$ is called a normal element of λ w.r.to μ if $\lambda(a) \ge \mu(y)\ \forall\ y \in Y$.

If λ be a fuzzy subgroup of G, then the identity element e of G is a normal element of λ w.r.to λ.

LEMMA 2.17: If (λ,τ) and (μ,τ_1) be fuzzy topological spaces and if a be a normal element of μ w.r.to λ, then the proper function $F: (\lambda,\tau) \longrightarrow (\mu,\tau_1)$ defined by

$$F(x,y) = \lambda(x) \text{ or } \emptyset \text{ according as } y = a \text{ or } y \ne a$$

is fuzzy continuous.

REMARK 2.18: If (λ,τ) and (μ,τ_1) be fuzzy topological spaces in the sense of Chakrabarty and Ahsanullah [1], then F, as defined above may not be fuzzy continuous.

DEFINITION 2.19: $\lambda\times\mu$: $X\times Y \longrightarrow I$ is defined by

$$(\lambda\times\mu)(x,y) = M(\lambda(x),\mu(y)) \ \forall\ (x,y) \in X\times Y.$$

DEFINITION 2.20: The proper function p_λ: $\lambda\times\mu \longrightarrow \lambda$, defined by

$p_\lambda((x,y),z) = (\lambda\times\mu)(x,y)$ or $\emptyset$ according as $z = x$ or $z \neq x$ $\forall\ x\ ,\ z \in X$ and $\forall\ y \in Y$ is said to be the fuzzy projection ma of $\lambda\times\mu$ into λ. Similarly one can define the fuzzy projection map p_μ of $\lambda\times\mu$ into μ.

LEMMA 2.21: Let $\delta \leq \lambda$, $\eta \leq \mu$. Then

(i) $p_\lambda^{-1}(\delta) = \delta\times\mu$, $p_\mu^{-1}(\eta) = \lambda\times\eta$;

(ii) $p_\lambda(\delta\times\eta) = \delta\cap k$ where $k = \sup.\{\eta(y);\ y \in Y\}$,

and $p_\mu(\delta\times\eta) = \eta\cap k_1$ where $k_1 = \sup.\{\delta(x);\ x \in X\}$.

REMARK 2.22: $p_\lambda(\lambda\times\mu)$ (or $p_\mu(\lambda\times\mu)$) may not be equal to λ (or μ). However if there exists a normal element of μ (or λ) w.r.to λ (or μ), then $p_\lambda(\lambda\times\mu) = \lambda$ (or $p_\mu(\lambda\times\mu) = \mu$).

3. PRODUCT FUZZY TOPOLOGY

Let (λ,τ) and (μ,τ_1) be two fuzzy topological spaces.

DEFINITION 3.1: The collection $\mathcal{B} = \{\gamma\times\eta;\ \gamma \in \tau, \eta \in \tau_1\}$ forms an open base of a fuzzy topology in $\lambda\times\mu$. The fuzzy topology in $\lambda\times\mu$, induced by $\mathcal{B}$ is called the product fuzzy topology of τ and τ_1 and is denoted by $\tau\times\tau_1$. The fuzzy topological space $(\lambda\times\mu,\tau\times\tau_1)$ is called the product of the fuzzy topological spaces (λ,τ) and (μ,τ_1).

THEOREM 3.2: p_λ : $(\lambda\times\mu,\tau\times\tau_1) \longrightarrow (\lambda,\tau)$ and p_μ: $(\lambda\times\mu,\tau\times\tau_1) \longrightarrow (\mu,\tau_1)$ are fuzzy continuous and fuzzy open. $\tau\times\tau_1$ is the smallest fuzzy topology in $\lambda\times\mu$ w.r.to which p_λ and p_μ are fuzzy continuous.

THEOREM 3.3: If a be a normal element of μ w.r.to λ, then the proper function F_α: $(\lambda,\tau) \longrightarrow (\lambda\times\mu,\tau\times\tau_1)$ defined by

$$F_\alpha(x,(x_1,y_1)) = \lambda(x) \text{ or } 0 \text{ according as } (x_1,y_1) = (x,a) \text{ or } (x_1,y_1) \neq (x,a)$$

is fuzzy continuous.

THEOREM 3.4: If (λ,τ) and (μ,τ_1) be fuzzy Hausdorff, so also is $(\lambda\times\mu,\tau\times\tau_1)$.

REMARK 3.5: The following example shows that the converse of Theorem 3.4 is not true.

EXAMPLE 3.6: Let $X = \{x,y\}$, $Y = \{a,b\}$. Let $\lambda = (0.4,0.5)$, $\mu = (0.3,0.2)$. Let $\mathcal{B}_1 = \{(0.4,0), (0,0.4)\}$, $\mathcal{B}_2 = \{(0.3,0), (0,0.2)\}$. Let $\tau = \tau(\mathcal{B}_1)$, $\tau_1 = \tau(\mathcal{B}_2)$. Then $(\lambda\times\mu,\tau\times\tau_1)$ is fuzzy Hausdorff but (λ,τ) is not fuzzy Hausdorff.

However we have the following result:

THEOREM 3.7: If there exists a normal element of μ(or λ) w.r.to λ (or μ), then the fuzzy Hausdorffness of $(\lambda\times\mu,\tau\times\tau_1)$ implies the fuzzy Hausdorffness of (λ,τ) (or (μ,τ_1)).

REMARK 3.8: The product of two fuzzy compact spaces may not be fuzzy compact as shown by

EXAMPLE 3.9: Let $X = \{a,b\}$, $Y = \{x_1,x_2,\ldots,x_n,\ldots\}$. Then $X\times Y = \{(a,x_1),(b,x_1),\ldots, (a,x_n),(b,x_n),\ldots\}$. Let $\lambda = (0.4,0.3)$, $\mu = (0.6,0.5,0.5,\ldots,0.5,\ldots)$. Let $\tau = \{\lambda\cap k;\ k \in I\}$ and $\tau_1 = \tau(\mathcal{B})$ where $\mathcal{B}$ is the collection of all fuzzy subsets G_α of μ of the form $G_\alpha(x_n) = 0$ or 0.4 for all n. Then (λ,τ) and (μ,τ_1) are both fuzzy compact but $(\lambda\times\mu,\tau\times\tau_1)$ is not fuzzy compact.

THEOREM 3.10: If there exists a normal element of μ(or λ) w.r.to λ (or μ), then the fuzzy compactness of $(\lambda\times\mu,\tau\times\tau_1)$

implies the fuzzy compactness of (λ,τ) (or (μ,τ_1)).

REMARK 3.11: The following examples will show that neither the fuzzy connectedness of (λ,τ) and (μ,τ_1) implies the fuzzy connectedness of $(\lambda\times\mu,\tau\times\tau_1)$ nor the fuzzy connectedness of $(\lambda\times\mu,\tau\times\tau_1)$ implies the fuzzy connectedness of (λ,τ) and (μ,τ_1).

EXAMPLE 3.12: Let $X = \{x,y\}$, $Y = \{a,b\}$. Let $\lambda = (0.3,0.4)$, $\mu = (0.4,1)$. Let $\tau = \{\lambda\cap k;\ k \in I\}$, $\tau_1 = \tau(\mathcal{B})$ where $\mathcal{B} = \{(0.4,0), (0,0.45)\}$. Then (λ,τ) and (μ,τ_1) are fuzzy connected. But since $(0.3,0.4)\times(0.4,0)$ and $(0.3,0.4)\times(0,0.45)$ are two disjoint members of $\tau\times\tau_1$ whose union is the fuzzy clopen set $\lambda\times\mu$, $(\lambda\times\mu,\tau\times\tau_1)$ is not fuzzy connected.

EXAMPLE 3.13: Let $X = \{a,b\}$, $Y = \{x,y\}$. Then $X\times Y = \{(a,x),(a,y),(b,x),(b,y)\}$. Let $\lambda = (0.5,0.7)$, $\mu = (0.4,0.6)$. Then $\lambda\times\mu = (0.4,0.5,0.4,0.6)$. Let $\tau = \tau(\mathcal{B})$ where $\mathcal{B} = \{(0.5,0.5),(0,0.2),(0.3,0.2),(0.2,0.3)\}$ and $\tau_1 = \{\mu\cap k;\ k \in I\}$. Then $(\lambda\times\mu,\tau\times\tau_1)$ is fuzzy connected but (λ,τ) is not fuzzy connected.

THEOREM 3.14: Let (λ_i,τ_i) and (μ_i,σ_i) be fuzzy topological spaces and F_i: $(\lambda_i,\tau_i) \longrightarrow (\mu_i,\sigma_i)$ be fuzzy continuous proper functions for $i = 1,2,\ldots,n$. Then the proper function $F = \prod F_i: \prod \lambda_i \longrightarrow \prod \mu_i$, defined by

$F((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \prod \lambda_i(x_1,\ldots,x_n)$ or 0 according as $(y_1,\ldots,y_n) = (y_{10},\ldots,y_{n0})$ or $(y_1,\ldots,y_n) \neq (y_{10},\ldots,y_{n0})$ is also fuzzy continuous.

4. FUZZY TOPOLOGICAL GROUPS:

Let G be a group with e as the identity element.

Let λ be a fuzzy subgroup of G. Let the proper functions $\lambda_p, \lambda_{pi}, \lambda_{ip}: \lambda\times\lambda \longrightarrow \lambda$ and $\lambda_i, \lambda_{ra}, \lambda_{la}: \lambda \longrightarrow \lambda$ be defined by

$\lambda_p((x,y),z) = (\lambda\times\lambda)(x,y)$ or $\emptyset$ according as $z = xy$ or $z \neq xy$,

$\lambda_{pi}((x,y),z) = (\lambda\times\lambda)(x,y)$ or $\emptyset$ according as $z = xy^{-1}$ or $z \neq xy^{-1}$

$\lambda_{ip}((x,y),z) = (\lambda\times\lambda)(x,y)$ or $\emptyset$ according as $z = x^{-1}y$ or $z \neq x^{-1}y$

$\lambda_i(x,y) = \lambda(x)$ or $\emptyset$ according as $y = x^{-1}$ or $y \neq x^{-1}$;

$\lambda_{ra}(x,y) = \lambda(x)$ or $\emptyset$ according as $y = xa$ or $y \neq xa$,

$\lambda_{la}(x,y) = \lambda(x)$ or $\emptyset$ according as $y = ax$ or $y \neq ax$

for all $x,y,z \in G$.

DEFINITION 4.1: A fuzzy topology τ on λ is called a fuzzy group topology if the proper functions $\lambda_p: (\lambda\times\lambda, \tau\times\tau) \longrightarrow (\lambda,\tau)$ and $\lambda_i: (\lambda,\tau) \longrightarrow (\lambda,\tau)$ are fuzzy continuous. The pair (λ,τ) is said to be a fuzzy topological group if τ be a fuzzy group topology on λ.

THEOREM 4.2: A fuzzy topology τ on λ is a fuzzy group topology iff the proper function λ_{pi}(or λ_{ip}): $(\lambda\times\lambda, \tau\times\tau) \longrightarrow (\lambda,\tau)$ is fuzzy continuous.

THEOREM 4.3: If (λ,τ) be a fuzzy topological group and if a be a normal element of λ w.r.to λ, then λ_{ra}, λ_{la} are fuzzy homeomorphisms of (λ,τ) onto itself.

Let λ and μ be two fuzzy subgroups of two groups G and H resoectively.

DEFINITION 4.4: A proper function $F: \lambda \longrightarrow \mu$ is said to be a fuzzy homomorphism if $F(x,y) = \lambda(x)$, $F(z,w) = \lambda(z)$ imply $F(xz,yw) = \lambda(xz)$ for all $x,y,z,w \in G$.

LEMMA 4.5: Let $F: \lambda \longrightarrow \mu$ be a fuzzy homomorphism. Then

(i) $F^{-1}(\mu)$ is a fuzzy subgroup of G,

(ii) $F(\lambda)$ is a fuzzy subgroup of H if F be injective.

THEOREM 4.6: Let F: $\lambda \longrightarrow \mu$ be a fuzzy homomorphism. If σ be a fuzzy group topology on μ, then $\tau = \{F^{-1}(\delta);\ \delta \in \sigma\}$ is a fuzzy group topology on λ.

THEOREM 4.7: Let F: $\lambda \longrightarrow \mu$ be an injective fuzzy homomorphism. If τ be a fuzzy group topology on λ, then $\sigma = \{\delta \leq \mu;\ F^{-1}(\delta) \in \tau\}$ is a fuzzy group topology on $F(\lambda)$.

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