The λ-additive Fuzzy Measure and Fubini Theorem

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Abstract: In this paper, we will discuss the integrals of λ-additive fuzzy measure on product space X×Y, and the Fubini theorem is proved.

Keywords: λ-additive fuzzy measure, fuzzy integral, Fubini theorem.

1. Introduction

In 1974, Sugeno [1] first introduced the notions of fuzzy measures and fuzzy integral. Afterwards, Sugeno [2] added a further axiom to fuzzy measure, i. e. λ-additivity. More details concerning these special fuzzy measure can be found in the papers by Kruse [2,3], Banon[5], Hua[7]. In1982, Kruse [3] proved that there exists a relationship between probability measure and λ-additive fuzzy measure. This relationship is used to definite a so called "fuzzy integral" of a fuzzy event with respect to a λ-additive fuzzy measure[4], which is a proper tool to express fuzzy expectations.

Some properties of λ-additive fuzzy measure have been investigated in [6,7,8,], in the present paper, continuing Sugeno's work on λ-additive fuzzy measures and Kruse's work on fuzzy integrals, in section 3, we first give a λ-additive fuzzy measure on product space, further, the Fubini theorem is proved.

2. Preliminaries

Let $(X, \mathscr{A})$ be a measurable space, a λ-additive fuzzy measure on $(X, \mathscr{A})$ in Sugeno's sense [2] is a set function

$g_\lambda : \mathscr{A} \to [0,1]$

which is continuous with respect to monotone sequences of sets and satisfying

(1) $g_\lambda(\varnothing)=0 \qquad g_\lambda(X)=1$

(2) $A, B \in \mathscr{A}, A \cap B = \varnothing, \Rightarrow g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda \cdot g_\lambda(A) \cdot g_\lambda(B)$

where $\lambda \in (-1, \infty)$. If $\lambda \neq 0$, by using the transformation

$$H(x) = \log_{1+\lambda}(1+\lambda x) \tag{2.1}$$

then

$$g^* = H(g_\lambda) \tag{2.2}$$

is a probability measure on $(X, \mathscr{A})$, and a fuzzy integral is defined like

$$\int_A f\, dg_\lambda = H^{-1}[\int_A H(f) dg^*] \tag{2.3}$$

where $f: X \to [0,1]$ be a $\mathscr{A}$-measurable function and $A \in \mathscr{A}$. Properties of fuzzy integral can be found in [4,7].

3. g_λ-integral and Fubini theorem on product space

Let X, Y be two arbitrary nonempty sets, and $X \times Y$ be their Cartesian product.

Definition 3.1 Let $A \subset X \times Y$, for any $x \in X$, we will call the set $A_x = \{y: (x, y) \in A\}$ a X-section determined by x. The concept of a

Y-section determined by $y \in Y$ is defined similarly by

$A_y = \{x: (x,y) \in A\}$.

Lemma 3.1[9]: If $A \subset X \times Y$, $B \subset X \times Y$, $A^{(n)} \subset X \times Y$, $n=1, 2, \cdots$, then

$$A \cap B = \emptyset \Rightarrow A_x \cap B_x = \emptyset,\ A_y \cap B_y = \emptyset \tag{3.1}$$

$$A \supset B \Rightarrow A_x \supset B_x, A_y \supset B_y \tag{3.2}$$

$$A = \bigcup_n A^{(n)} \Rightarrow A_x = \bigcup_n A_x^{(n)}, A_y = \bigcup_n A_y^{(n)} \tag{3.3}$$

$$A = \bigcap A^{(n)} \Rightarrow A_x = \bigcap A_x^{(n)}, A_y = \bigcap A_y^{(n)} \tag{3.4}$$

Lemma 3.2[9]: Let $(X, \mathscr{A}_1)$ and $(Y, \mathscr{A}_2)$ are measurable spaces, if $A \in \mathscr{A}_1 \times \mathscr{A}_2$, then for any $x \in X$, $A_x \in \mathscr{A}_2$, and for any $y \in Y$, $A_y \in \mathscr{A}_1$.

Definition 3.2: Let $f(x,y)$ is a function on $X \times Y$, for any $x \in X$, we will call the function f_x, defined by

$$f_x(y) = f(x,y) \qquad y \in Y \tag{3.5}$$

an X-section of f determined by x. The concept of a Y-section of f determined by $y \in Y$ is defined similarly by

$$f_y(x) = f(x,y) \qquad x \in X \tag{3.6}$$

Lemma 3.3[9]: If $f(x,y)$ is a measurable function on $(X \times Y, \mathscr{A}_1 \times \mathscr{A}_2)$, then for any $x \in X$, $f_x(y)$ is a measurable function on $(Y, \mathscr{A}_2)$. For any $y \in Y$, $f_y(x)$ is a measurable function on $(X, \mathscr{A}_1)$.

Lemma 3.4[9]: Let $f(x,y)$ is a nonnegative measurable function on $(X \times Y, \mathscr{A}_1 \times \mathscr{A}_2)$, μ_1 and μ_2 are σ-finite measures on $(X, \mathscr{A}_1)$ and $(Y, \mathscr{A}_2)$ respectively, then

$$\int_X f(x,y)d\,\mu_1(x) \qquad \left(\int_Y f(x,y)d\,\mu_2(y) \right) \tag{3.7}$$

is a measurable function on $(Y, \mathscr{A}_2)$ $((X, \mathscr{A}_1))$. Specially, for every $A \in \mathscr{A}_1 \times \mathscr{A}_2$, $\mu_1(A_y)$ $(\mu_2(A_x))$ is a measurable function on

$(Y, \mathscr{A}_2)$ ($(X, \mathscr{A}_1)$).

Lemma3.5[9]: Suppose $(X, \mathscr{A}_1, \mu_1,)$ and $(Y, \mathscr{A}_2, \mu_2)$ are two σ-finite measure spaces, and let

$$\mu(A) \triangleq \int_X \mu_2(A_x) d\mu_1(x) \qquad A \in \mathscr{A}_1 \times \mathscr{A}_2 \tag{3.8}$$

or

$$\mu(A) \triangleq \int_Y \mu_1(A_y) d\mu_2(y) \qquad A \in \mathscr{A}_1 \times \mathscr{A}_2 \tag{3.9}$$

Then, $\mu(\cdot)$ is an unique σ-finite measure on $(X \times Y, \mathscr{A}_1 \times \mathscr{A}_2)$, such that

$$\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2) \qquad A_i \in \mathscr{A}_i \ (i=1,2)$$

Theorem3.1: Let $(X, \mathscr{A}_1)$ and $(Y, \mathscr{A}_2)$ are two measurable spaces, $f(x,y): X \times Y \to [0,1]$ is a $\mathscr{A}_1 \times \mathscr{A}_2$-measurable function, g_1 and g_2 are λ-additive fuzzy measure on $(X, \mathscr{A}_1)$ and $(Y, \mathscr{A}_2)$ respectively. Then

$$\int_X f(x,y) dg_1(x) \qquad \left(\int_Y f(x,y) dg_2(y) \right) \tag{3.10}$$

is a measurable function on $(Y, \mathscr{A}_2)$, ($(X, \mathscr{A}_1)$). Specially, for any $E \in \mathscr{A}_1 \times \mathscr{A}_2$, $g_1(E_y)$ ($g_2(E_x)$)is a measurable function on $(Y, \mathscr{A}_2)$ ($(X, \mathscr{A}_1)$).

Proof: From lemma3.3, for any $y \in Y$, $f_y(x) = f(x,y)$ is a $\mathscr{A}_1$-measurable function, and $0 \leqslant f_y(x) \leqslant 1$. Thus

$$\int_X f(x,y) dg_1 = \int_X f_y(x)\, dg_1 = H^{-1}\left[\int_X H(f_y(x)\, dg_1^*\right]$$

Since $H(x)$ is a continous function, from lemma3.4, $\int_X f(x,y) dg_1$ is a $\mathscr{A}_2$-measurable function.

Specially, for any $E \in \mathscr{A}_1 \times \mathscr{A}_2$, if we make $f(x,y) = X_E(x,y)$, where $X_E(x,y)$ is a characteristic function of E, then

$$\int_X f(x,y) dg_1 = \int_X X_E(x,y) dg_1 = \int_X X_{E_y}(x) dg_1$$

$$=H^{-1}\left[\int_X H(X_{E_y}(x))dg_1^*\right]$$
$$=H^{-1}[g_1^*(E_y)]=g_1(E_y)$$

is $\mathscr{A}_2$-measurable function.

Theorem3. 2: Suppose $(X,\mathscr{A}_1,g_1)$ and $(Y,\mathscr{A}_2,g_2)$ are two λ-additive fuzzy measure spaces, for any $E\in\mathscr{A}_1\times\mathscr{A}_2$, let

$$g(E)\triangleq\int_X g_2(E_X)dg_1(x) \tag{3.11}$$

or

$$g(E)\triangleq\int_Y g_1(E_y)dg_2(y) \tag{3.12}$$

then $g(\cdot)$ is a λ-additive fuzzy measure on $(X\times Y,\ \mathscr{A}_1\times\mathscr{A}_2)$.

Proof: For any $E\in\mathscr{A}_1\times\mathscr{A}_2$, first, we have

$$g(E)=\int_X g_2(E_x)dg_1(x)=H^{-1}\left[\int_X H[g_2(E_x)]dg_1^*\right]$$
$$=H^{-1}\left[\int_X g_2^*(E_x)dg_1^*\right]=H^{-1}\left[\int_Y g_1^*(E_y)dg_2^*\right]=\int_Y g_1(E_y)dg_2(y)$$

It is easy to see, $\int_X g_2^*(E_x)dg_1^*$ is a probability measure, therefore, from(2. 2), $H^{-1}\left[\int_X g_2^*(E_x)dg_1^*\right]$ is a λ-additive fuzzy measure.

Corollary : Suppose $g(\cdot)$ is a λ-additive fuzzy measure on $(X\times Y,\mathscr{A}_1\times\mathscr{A}_2)$. Then for any $E_1\in\mathscr{A}_1$, $E_2\in\mathscr{A}_2$

$$H[g(E_1\times E_2)]=H[g_1(E_1)]\cdot H[g_2(E_2)] \tag{3.15}$$

Proof : From theorem 3. 2, we have

$$H[g(E_1\times E_2)]=g^*(E_1\times E_2)=\int_X g_2^*(E_1\times E_2)_x\,dg_1^*$$
$$=\int_X g_2^*(E_2)\cdot X_{E_1}(x)dg_1^*=g_1^*(E_1)\cdot g_2^*(E_2)$$
$$=H[g_1(E_1)]\cdot H[g_2(E_2)]$$

Theorem3. 3 (Fubini) : Let $(X,\mathscr{A}_1,g_1)$, $(Y,\mathscr{A}_2,g_2)$ and $(X\times Y,\mathscr{A}_1\times\mathscr{A}_2,g)$ are λ-additive fuzzy measure spaces, $f(x,y):X\times Y\to[0,1]$ is $\mathscr{A}_1\times\mathscr{A}_2$-measurable function, then

$$\int_{X\times Y} f(x,y)\,dg=\int_X\left[\int_Y f(x,y)dg_2\right]dg_1=\int_Y\left[\int_X f(x,y)dg_1\right]dg_2$$

Proof : By lemma3. 3, lemma3. 4, theorem3. 1 and the Fubini theorem with respect to probability measure, we have

$$\begin{aligned}\int_X\left(\int_Y f(x,y)dg_2\right)dg_1 &=\int_X H^{-1}\left[\int_Y H(f)dg_2^*\right]dg_1\\ &=H^{-1}\left[\int_X H\left[H^{-1}\left[\int_Y H(f)dg_2^*\right]\right]dg_1^*\right]\\ &=H^{-1}\left[\int_X\left[\int_Y H(f)\, dg_2^*\right]dg_1^*\right]\\ &=H^{-1}\left[\int_Y\left[\int_X H(f)dg_1^*\right]dg_2^*\right]\\ &=\int_Y\left[\int_X fdg_1\right]dg_2.\end{aligned}$$

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