A Chainability Criterion for Fuzzy Implications

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Abstract

Fuzzy implications, also known as fuzzy IF–THEN rules, play an important role in applications of fuzzy set based techniques e.g. in automatic control. For such implications the problem has been discussed whether two of them, being connected in the usual way with the succedent of the first rule and the antecedent of the second one identical, can be "chained" in the usual way as described by the traditional rule of syllogism in formal logic.

Here a quite general necessary and sufficient condition is given under which this type of chaining has the desired properties, and is applied to one of the standard types of coding of fuzzy implications.

1 The chainability of fuzzy implications

What is usually called a *fuzzy implication* is some particular conditional statement given as an **if–then** rule of the form

$$\textsf{if} \quad x \text{ is } A \qquad \textsf{then} \quad y \text{ is } B. \tag{1}$$

Here A and B are fuzzy sets which themselves are interpreted as (fuzzy or linguistic) values of some variables x resp. y. The fuzzy set A shall also be called the *antecedent fuzzy datum* and the fuzzy set B accordingly the *consequent fuzzy datum* of this fuzzy implication (1).

Having given two such fuzzy implications of the particular form **if** x is A **then** y is B and **if** y is B **then** z is C, one may pose the problem, whether from these two fuzzy implications a third one **if** x is A **then** z is C *follows*. This problem was discussed e.g. in [2]. The corresponding problem related to the fuzzy method-of-cases was treated e.g. in [6], [7].

We follow the common usage to read the fuzzy implications in the same way as is done with the control rules of a fuzzy controller: as fuzzy relations. That means that the fuzzy implication (1) has to be transformed into - or *coded* by - a fuzzy relation R. And this fuzzy relation R obviously has to be determined by the fuzzy antecedent and consequent data A, B explicitely mentioned in this fuzzy implication.

Definition 1 *A* coding procedure Θ *for fuzzy implications of the type*

$$\textsf{if} \quad x \textit{ is } A \quad \textsf{then} \quad y \textit{ is } B \tag{2}$$

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with $A \in \mathbb{F}(\mathcal{X})$ *and* $B \in \mathbb{F}(\mathcal{Y})$ *is an operator*

$$\Theta : \mathbb{F}(\mathcal{X}) \times \mathbb{F}(\mathcal{Y}) \to \mathbb{F}(\mathcal{X} \times \mathcal{Y}) \tag{3}$$

which connects fuzzy relations with given pairs of fuzzy sets.

For chainability, however, one has not only to look at such a particular coding procedure. Fuzzy implications which shall be taken into consideration for chaining also have to be suitably related.

Definition 2 *Two fuzzy implications*

$$\begin{array}{llll} \textsf{if} & x_1 \ is \ A_1 & \textsf{then} & y_1 \ is \ B_1, \\ \textsf{if} & x_2 \ is \ A_2 & \textsf{then} & y_2 \ is \ B_2 \end{array}$$

are called connected *iff either one has* $y_1 = x_2$ *and* $B_1 = A_2$ *or one has* $y_2 = x_1$ *and* $B_2 = A_1$.

As we are interested to discuss the chainability in our setting of fuzzy implications, we also need to consider the result of chaining two connected implications. To do this effectively, let us extend our terminology a bit.

Definition 3 *Having given two connected fuzzy implications*

$$\begin{array}{llll} \textsf{if} & x \ is \ A & \textsf{then} & y \ is \ B, \\ \textsf{if} & y \ is \ B & \textsf{then} & z \ is \ C \end{array}$$

by their chained fuzzy implication *we shall mean the fuzzy implication*

$$\textsf{if} \quad x \ is \ A \quad \textsf{then} \quad z \ is \ C. \tag{4}$$

In the case that one intends to discuss two such connected fuzzy implications and to compare them with their chained fuzzy implication the coding of fuzzy implications by fuzzy relations means that three fuzzy relations R, S and T are given and that the fuzzy relations R, S and T have to be compared in some suitable sense.

What now seems to be a reasonable understanding of comparison here?

It is obvious from the use in fuzzy control, that fuzzy implications – as well as the former control rules – in the context of information processing have to act as tools to transform some given piece of information into another piece of information. Or to put it more formally: fuzzy implications have to be understood in such a way that they are tools to transform a given fuzzy set into another one.

From this point of view, connected fuzzy implications then transform a first piece of information into a second one, and this second piece of information further into a third one. But now there is a quite natural understanding of what it intuitively shall mean that *two fuzzy implications are chainable*: The result of transforming a given piece of information according to two connected fuzzy implications should be the same as transforming this piece of information according to the chained fuzzy implication. And this should be the case for all given pieces of information.

To make the intuition behind the idea of information transfer via fuzzy implications precise, we furthermore have to fix the method how to determine the piece of information which some particular fuzzy implications yields if applied to some particular piece of information, i.e. we have to determine a method which produces a fuzzy set out of a given fuzzy relation and a given fuzzy set.

This method, again, has to be understood as some operator mapping the cartesian product $\mathbb{F}(\mathcal{X} \times \mathcal{Y}) \times \mathbb{F}(\mathcal{X})$ into the class $\mathbb{F}(\mathcal{Y})$ of fuzzy subsets of $\mathcal{Y}$. Here, however, we shall restrict our considerations to the particular case that this operator is determined by the generalisation

$$\mu_B(y) = \sup_{x \in \mathcal{X}} \boldsymbol{t}(\mu_A(x), \mu_R(x,y)), \tag{5}$$

of the "compositional rule of inference" with the t-norm $\boldsymbol{t}$ instead of the commonly used minimum operator $\wedge$. For the fuzzy set B determined in this way we write as usual also $A \circ_{\boldsymbol{t}} R$. The reason for this restriction, if compared with the earlier approach toward the coding procedure of fuzzy implications by fuzzy relations, actually lies in the simple fact that in the present applications of fuzzy implications and fuzzy control rules e.g. in automated control and expert systems it is common usage to refer to the compositional rule of inference for combining fuzzy sets with fuzzy relations, but there exists a much wider field of approaches toward coding fuzzy implications or control rules by fuzzy relations.

Definition 4 *Two particular connected fuzzy implications*

if x *is* A then y *is* B,
if y *is* B then z *is* C

are $\boldsymbol{t}$-chainable *(w.r.t. the coding procedure* Θ*) into the fuzzy implication*

if x *is* A then z *is* C.

iff for all fuzzy sets $A' \in \mathbb{F}(\mathcal{X})$ *it holds true that*

$$(A' \circ_{\boldsymbol{t}} \Theta(A,B)) \circ_{\boldsymbol{t}} \Theta(B,C) = A' \circ_{\boldsymbol{t}} \Theta(A,C). \tag{6}$$

And the coding procedure Θ *has the* $\boldsymbol{t}$-chainability property *iff any two connected fuzzy implications are* $\boldsymbol{t}$*-chainable w.r.t. this coding procedure* Θ*.*

Both these notions of the $\boldsymbol{t}$-chainability of two fuzzy implications and of the $\boldsymbol{t}$-chainability property of a coding procedure obviously cover the intuitions we discussed earlier in this paper. Furthermore there is a nice characterisation of the $\boldsymbol{t}$-chainability property of a coding procedure Θ which does not refer to the actual inputs but only uses the "fuzzy data" explicitly given in the fuzzy implications.

Theorem 1 *Let* $\boldsymbol{t}$ *be a left continuous t-norm. A coding procedure* Θ *has the* $\boldsymbol{t}$*-chainability property iff for all fuzzy sets* $A \in \mathbb{F}(\mathcal{X}), B \in \mathbb{F}(\mathcal{Y}), C \in \mathbb{F}(\mathcal{Z})$ *one has*

$$\Theta(A,B) \circ_{\boldsymbol{t}} \Theta(B,C) = \Theta(A,C) \tag{7}$$

w.r.t. the $\boldsymbol{t}$*-based fuzzy relational product* $\circ_{\boldsymbol{t}}$*.*

A closer inspection of the proof shows that the arguments also apply in the particular case that the fuzzy sets A, B, C are fixed from the very beginning. That means one also has the

Corollary 2 *Any two particular connected fuzzy implications*

$$\begin{array}{llll} \text{if} & x \textit{ is } A & \text{then} & y \textit{ is } B, \\ \text{if} & y \textit{ is } B & \text{then} & z \textit{ is } C \end{array}$$

are $\boldsymbol{t}$*-chainable iff one has*

$$\Theta(A,B) \circ_{\boldsymbol{t}} \Theta(B,C) = \Theta(A,C).$$

2 Chainability for the cartesian product coding

The earliest way of coding fuzzy implications or fuzzy (control) rules, and still one which in applications is often used, goes back to [5], cf. also [1]. The coding procedure is defined via

$$\Theta_0(A,B) =_{\text{def}} A \times_{\boldsymbol{t}_1} B \tag{8}$$

with reference to a t-norm $\boldsymbol{t}_1$ and thus equivalently characterised as

$$C := \Theta_0(A,B): \;\; \mu_C(a,b) = \boldsymbol{t}_1(\mu_A(a), \mu_B(b)).$$

In the following we suppose that this t-norm $\boldsymbol{t}_1$ is *left continuous*, as it shall be the case with the t-norm $\boldsymbol{t}$ which is used in the (generalised) compositional rule of inference (5).

Proposition 3 *Let a pair of connected fuzzy implications*

$$\begin{array}{llll} \text{if} & x \textit{ is } A & \text{then} & y \textit{ is } B, \\ \text{if} & y \textit{ is } B & \text{then} & z \textit{ is } C \end{array}$$

be given and consider the cartesian product coding (8) based on the t-norm $\boldsymbol{t}_1$*. Then a necessary condition for the* $\boldsymbol{t}$*-chainability of this pair of fuzzy implications is that one has*

$$\text{hgt}\,(A) \wedge \text{hgt}\,(C) \leq \text{hgt}\,(B). \tag{9}$$

Furthermore, one proves the following interesting fact.

Theorem 4 *If the* $\boldsymbol{t}$*-chainability property holds true for the coding procedure (8) by the* $\boldsymbol{t}_1$*-based cartesian product, then the t-norms* $\boldsymbol{t}_1$ *and* $\boldsymbol{t}$ *have to be identical:* $\boldsymbol{t}_1 = \boldsymbol{t}$.

The result of Proposition 3 shows, that in the case of $\text{hgt}\,(B) < 1$ one always will be able to find particular connected fuzzy implications – connected "via" the fuzzy datum B – which are not $\boldsymbol{t}$-chainable. On the other hand it is almost common usage in fuzzy control applications to suppose that the fuzzy input and output values which are explicitely mentioned in the control rules are normal fuzzy sets. Therefore a restriction of our considerations to normal fuzzy sets is reasonable.

Definition 5 *A coding procedure Θ has the* normal t-chainability property *iff any two connected fuzzy implications with only normal fuzzy sets as antecedent and consequent fuzzy data are t-chainable w.r.t. this coding procedure Θ.*

With this modified notion of chainability property we get the following general result.

Theorem 5 *The coding procedure (8) by the t-based fuzzy cartesian product has the normal t-chainability property.*

The proof of this theorem yields even a bit more which, nevertheless, is not as interesting as the theorem itself. The proof of the following corollary is obvious.

Corollary 6 *A necessary and sufficient condition for the t-chainability of two particular fuzzy implications w.r.t. the cartesian product coding procedure $\Theta_0(A, B) = A \times_t B$ is, that the connecting fuzzy datum is a normal fuzzy set.*

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