Book Review of

"Non-Additive Measure and Integral"

by Dieter DENNEBERG

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The aim of this short book is to explain the mathematical foundations of Choquet's integral. It is an important contribution to the literature of non-additive probability. The idea that set-functions representing uncertainty need not be additive has been revived in the last thirty years or so with the emergence of fuzzy measures, belief functions, possibility measures, and the like. It has become notorious that non-additivity in probability started with Bernoulli in the 17th century, but that this view has been forgotten for some 200 years, with the development of additive probability and measure theory. Among the various non-additive theories of uncertainty available to-date, the least deviant and altogether the most general one considers probability bounds derived from a convex set of probability measures. It is a natural relaxation of the Bayesian credo of a unique probability measure being both necessary and sufficient to represent degrees of belief. Such probability bounds are monotonic set-functions, that satisfy relaxed versions of addivity (like super-additivity or subadditivity). The oldest extensive study of such set-functions paradoxically appeared in a very long paper devoted to electricity problems by the French mathematician Gustave Choquet [1], and monotonic set-functions were there called "capacities". These set-functions were rediscovered in the field of economics by David Schmeidler who was looking for representation theorems in utility theory when axioms proposed by Savage are relaxed. Independently, monotonic set-functions were proposed by Sugeno [10] in connection with fuzzy set theory, in order to represent degrees of belief. Actually, in subsequent work, these set-functions were used by Sugeno as generalized weights in multiple criteria aggregation procedures (e.g., Grabisch et al. [4]).

There are two types of integral-like constructs based on monotonic set-functions: Sugeno's integral which is ordinal, and Choquet's integral which generalizes Lebesgue integral over to non-additive set-functions. The book by

Denneberg deals with the latter integral. The main thrust of the book is to very clearly show that the notion of integral does not require additivity. The mode of construction of the non-additive integral is in two steps. Consider a set-function g (the non-additive measure) on a set Ω, taking values in the positive extended real line, and a function X from Ω to the real line. First it is always possible to define a distribution function for g, with respect to X. The only requirement is the monotonicity of g with respect to set-inclusion. The distribution of g with respect to X takes the form of a decreasing real function as shown in Chapter 4. The second step is to compute the integral of the pseudo-inverse of the distribution over the range of the non-additive measure. This is done by means of a Riemann integral.

The book is structured as follows. It starts at a very elementary level by a refresher on Riemann integration of monotone functions on intervals. Then it introduces various types of non-additive measures, including inner and outer extensions thereof; precise results on Caratheodory measurability are also recalled, i.e., conditions under which the additive part of a non-additive set-function is defined over a σ-algebra. Chapter 3 recalls the interactions between measurability and topology, examplified by the construction of Lebesgue-Stieltjes measures. The book then proceeds to the study of distribution functions in the non-additive case, and introduces the important notion of comonotonicity of functions. The importance of this notion, introduced by Schmeidler, is due to the fact that the pseudo-inverse of the distribution of a non-additive measure with respect to the sum of two comonotonic functions is the sum of the pseudo-inverses of the distributions with respect to each function. Two types of integral of a real-valued function with respect to a non-additive measure are defined, the asymmetric and the symmetric integrals respectively. It is shown that the subadditivity property of a non-additive measure carries over to the asymmetric integral, and that the asymmetric integral is additive for comonotonic functions. The symmetric integral is another possible extension of the usual integral for additive measures. It coincides with the asymmetric integral for positive functions but does not obey comonotonic additivity. There is a chapter on convergence properties of the non-additive integrals, and a long chapter devoted to "almost everywhere" true properties with respect to a non-additive measure, norms of functions, and null-sets. A short but incisive chapter shows that an important subclass of non-additive measures can be interpreted as probability bounds. The last chapters develop the notion of density of a non-additive measures, and some preliminary results on the extension of Fubini's theorem (on product spaces). Finally, results on the representation of functionals as non-additive integrals (a basic issue in the mathematics of decision theory) are presented.

It must be stressed that this book is a mathematical treatise and almost does not refer to the fields of applications where the presented concepts emerged. It is meant to present in a tutorial way the works of other researchers, and noticeably G. Choquet, G. Greco, D. Schmeidler and P. Wakker, although the author has also contributed to the field. And indeed the book has very high tutorial merits because it is almost self-contained and starts at a rather elementary level. Its reading can be highly beneficial to any student or researcher involved in decision theory including multiple criteria decision-making, but also artificial intelligence, and fuzzy set theory. The last point deserves some comments. For any fuzzy set-theorist reading this book, it becomes patent that the method used by the author for developing the non-additive integral is the level-cut method used for extending mathematical notions from sets to fuzzy sets (this is particularly clear on page 46, in Chapter 3). For instance the degree of belief in a fuzzy event following Smets [9] is a Choquet integral, as are the bounds of the interval that characterizes the mean-value of a fuzzy number (Dubois and Prade [3]). The additivity property for comonotonic functions is closely related to the extension principle for the addition of fuzzy numbers (for instance the mean value of the sum of two fuzzy numbers is the sum of their mean values as proved in [3]; this result becomes obvious noticing that the increasing (resp.: decreasing) parts of the membership functions of fuzzy numbers are comonotonic).

The book by Denneberg never discusses links between Choquet integral and the fuzzy set literature. For instance it apparently ignores the works of Murofushi and Sugeno [5, 6, 7]. Also, Choquet integral has a counterpart in an ordinal scale, Sugeno integral, on which there exists a mathematically-oriented monograph (Wang and Klir [10], see the book review [2]). Again, the author does not try any comparative discussion, although the term "fuzzy measure" appears on the back of the book. Yet one may think it a useful task to cross compare the works done on non-additive probability in an independent way in decision theory and fuzzy set theory. This is achieved in the more recent monograph by Grabisch et al. [4]. As for Denneberg's book, its ambition is clearly to serve as a text-book on Choquet's integral. It is obvious that this goal is fully reached, and that this book will become a landmark in measure theory, as offering a self-contained presentation of a concept of non-additive integral that until now was considered as exotic. Thanks to this book, non-additive probability will be taught to new generations of students, in mathematics, decision theory, statistics and artificial intelligence.

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