ON THE PSEUDO AND CONVERSE AUTOCONTINUITY OF FUZZY MEASURES

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Abstract: *The equivalence of the pseudo null-additivity and pseudo autocontinuity of fuzzy measures on S-compact spaces is proved. The converse autocontinuity is also investigated.*

Keywords: *Fuzzy measures, pseudo nulll-additivity, converse null-additivity, pseudo autocontinuity, converse autocontinuity, S-compact spaces.*

1 Introduction

The 'pseudo ' and 'converse' concepts of set functions are first proposed and discussed by Wang [3], Just as the autocontinuity, the pseudo and converse autocontinuities also play important roles in fuzzy measure theory. Sun [2] and Xing [5] have given a set of characterizations for pseudo autocontinuity, which are applied to the convergence in measure of sequence of measurable functions on fuzzy measure spaces. We have investigated the uniform converse and pseudo autocontinuities of finite fuzzy measures [1]. In this paper, we further show the equivalences of the pseudo null-additivity and pseudo autocontinuity, the converse null-additivity and converse autocontinuity from below for fuzzy measures on S-compact spaces, respectively, which was introduced by Wang [4].

2 pseudo autocontinuity

Let X be a nonempty set, $\mathcal{F}$ a σ-algebra of subsets of X, and let set function $\mu : \mathcal{F} \to [0, +\infty]$ is a fuzzy measures. Unless stated otherwise, all the subsets are supposed to belong to $\mathcal{F}$.

Definition 1 *μ is said to be pseudo null-additive if $\mu(B \cap C) = \mu(C)$ whenever $B, C \in A \cap \mathcal{F}, \text{and} \mu(B) = \mu(A) < +\infty$; pseudo autocontinuous from above (resp. pseudo autocontinuous from below) if, for $\mu(A) < +\infty$ and $C \in A \cap \mathcal{F}$,*

$$\mu(B_n \cap A) \to \mu(A) \Rightarrow \mu((A - B_n) \cup C) \to \mu(C) \quad (\textit{resp.}\, \mu(B_n \cap C) \to \mu(C)).$$

Definition 2 *a measurable space $(X, \mathcal{F})$ is said to be S-compact space if, for any sequence $\{E_n\}_n$, there exists a subsequence $\{E_{n(k)}\}_k$ of $\{E_n\}_n$ such that*

$$\bigcap_{k=1}^{+\infty}\bigcup_{i=k}^{+\infty} E_{n(i)} = \overline{\lim}_k\, E_{n(k)} = \underline{\lim}_k\, E_{n(k)} = \bigcup_{k=1}^{+\infty}\bigcap_{i=k}^{+\infty} E_{n(i)}.$$

For example, if X is finite or infinite countable, then $(X, \mathcal{F})$ is S-compact. In the remaining part of this paper, we assume that $(X, \mathcal{F})$ is a S-compact space.

Theorem 1 *The pseudo null-additivity of μ is equal to the pseudo autocontinuty from below.*

Proof It is sufficient to prove that the pseudo null-additivity implies the pseudo autocontinuity from below.

Let $\mu(B_n \cap A) \to \mu(A) < +\infty$ and $C \subset A$. We can choose a subsequence $\{B_{n(k)}\}_k$ such that $\underline{\lim}_n \mu(C \cap B_n) = \lim_k \mu(C \cap B_{n(k)})$ and $\overline{\lim}_k\, B_{n(k)} = \underline{\lim}_k\, B_{n(k)}$. Then,

$$\begin{aligned} \mu(A) &= \lim_k \mu(B_{n(k)} \cap A) \\ &\leq \lim_k \mu(\bigcup_{i=k}^{+\infty} B_{n(i)} \cap A) = \mu(\overline{\lim}_k B_{n(k)} \cap A) \\ &= \mu(\underline{\lim}_k\, B_{n(k)} \cap A) \leq \mu(A). \end{aligned}$$

Thus, we have $\mu(A) = \mu(\overline{\lim}_k B_{n(k)} \cap A)$ and $\mu(\overline{\lim}_k B_{n(k)} \cap C) = \mu(C)$, by the pseudo null-additivity of μ. On the other hand,

$$\begin{aligned}
\mu(C) &\geq \overline{\lim}_n \mu(B_n \cap C) \\
&\geq \underline{\lim}_n \mu(B_n \cap C) = \lim_k \mu(B_{n(k)} \cap C) \\
&\geq \lim_k \mu(\bigcap_{i=k}^{+\infty} B_{n(i)} \cap C) = \mu(\underline{\lim}_k B_{n(k)} \cap C) \\
&= \mu(\overline{\lim}_k B_{n(k)} \cap C)
\end{aligned}$$

Therefore, $\mu(C) = \lim_n \mu(B_n \cap C)$. □

Similarly, we have the following result.

Theorem 2 *The pseudo null-additivity of μ is equal to the pseudo autocontinuous from above*

3 Converse autocontinuity

Definition 3 *μ is said to be converse null-additive if $\mu(A-B) = 0$ whenever $B \subset A$ and $\mu(B) = \mu(A) < +\infty$; converse autocontinuous from above (resp. converse autocontinuous from below) if $\mu(B_n \cup A) \to \mu(A) \Rightarrow \mu(B_n - A) \to 0$ (resp. $\mu(B_n \cap A) \to \mu(A) \Rightarrow \mu(A - B_n) \to 0$) whenever $\mu(A) < +\infty$.*

Theorem 3 *The converse null-additivity of μ is equal to converse autocontinuty from below.*

Proof It is sufficient to prove that the converse null-additivity implies the converse autocontinuity from below.

Let $\mu(B_n \cap A) \to \mu(A) < +\infty$. We take a subsequence $\{B_{n(k)}\}_k$ such that $\overline{\lim}_n \mu(A - B_n) = \lim_k \mu(A - B_{n(k)})$ and $\overline{\lim}_k B_{n(k)} = \underline{\lim}_k B_{n(k)}$. Then,

$$\begin{aligned}\mu(A) &= \lim_k \mu(B_{n(k)} \cap A) \leq \lim_k \mu(\bigcup_{i=k}^{+\infty} B_{n(i)} \cap A) \\ &= \mu(\overline{\lim}_k B_{n(k)} \cap A) = \mu(\underline{\lim}_k B_{n(k)} \cap A) \leq \mu(A),\end{aligned}$$

and, hence, $\mu(A - underline\lim_k B_{n(k)}) = 0$, by the converse null-additivity. On the other hand,

$$\begin{aligned}\overline{\lim}_n \mu(A - B_n) &= \lim_k \mu(A - B_{n(k)}) \leq \lim_k \mu(\bigcup_{i=k}^{+\infty} (A - B_{n(k)})) \\ &= \mu(\overline{\lim}_k (A - B_{n(k)})) = \mu(A - \underline{\lim}_k B_{n(k)})\end{aligned}$$

Then, $\lim_n \mu(A - B_n) = 0$. □

Example 1 Let $X = \{1, 2, \ldots\}, \mathcal{F} = \wp(X)$, and $m(E) = \sum_{i \in E} \frac{1}{2^i}$. Put

$$\mu(E) = \begin{cases} \frac{n}{2 \cdot (n+1)} & \text{if } E = \{n\}, n > 1; \\ 0 & \text{if } E = \emptyset \\ 1 + Card(E - \{1\}) \cdot m(E - \{1\}) & \text{otherwise.} \end{cases}$$

Then, μ is converse and pseudo null-additive, and, hence, it is pseudo autocontinuity. But, μ is not converse autocontinuous from above.

Definition 4 *μ is said to be strongly order continuous if $A_n \searrow A$ and $\mu(A) = 0$ implies* $\lim_n \mu(A_n) = 0$.

If μ is finite fuzzy measure, then it is strongly order continuous, but the converse inclusion is not true.

Theorem 4 *Let μ be strongly order continuous. Then, μ has the converse autocontinuity from above if and only if it has the null-additivity.*

Proof Let $\mu(B_n \cup A) \to \mu(A)$ and μ be converse null-additive. We can choose a subsequence $\{B_{n(k)}\}_k$ of $\{B_n\}_n$ such that $\overline{\lim}_n \mu(B_n - A) = \lim_k \mu(B_{n(k)} - A)$ and $\overline{\lim}_k B_{n(k)} = \underline{\lim}_k B_{n(k)}$. Hence, we have

$$\begin{aligned}\mu(A) &= \lim_k \mu(A \cup B_{n(k)}) \geq \lim_k \mu(\bigcap_{i=k}^{+\infty} B_{n(i)} \cup A) \\ &= \mu(\underline{\lim}_k B_{n(k)} \cup A) = \mu(\overline{\lim}_k B_{n(k)} \cup A) \geq \mu(A)\end{aligned}$$

and, hence, $\mu(\overline{\lim}_k B_{n(k)} - A) = 0$. Furthermore, we have

$$\begin{aligned}\overline{\lim}_n \mu(B_n - A) &= \lim_k \mu(B_{n(k)} - A) \\ &\leq \lim_k \mu(\bigcup_{i=k}^{+\infty} B_{n(i)} - A) = 0\end{aligned}$$

by the strong order continuity of μ. This shows that $\lim_n \mu(A - B_n) = 0$. □

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