THE H–VALUED FUZZY ORTHOGONAL MEASURES ON REGULAR FUZZY MEASURABLE SPACE

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Abstract: The concept of H–valued fuzzy orthogonal measure (abbr. HFOM) on regular fuzzy measurable space is proposed. The representation theorem of HFOM's is proved. The properties of the normal family of HFOM's and the weak convergence of HFOM's are also investigated.
Keywords: Hilbert space; fuzzy measure space; weak convergence.

1. Introduction

Throughout this paper, let X be a non–empty crisp set and F(X) the class of all fuzzy sets in X. The index function of the subset E of X is denoted by χ_E. Let Σ be a non–empty sub–class of F(X) and $\sigma(\Sigma)$ the σ–algebra generated by the class of sets $\{\tilde{A}_{-1}(B)$: $B \in B_0$, $\tilde{A} \in \Sigma\}$, where B_0 is the Borel algebra of [0, 1]. We will call $\sigma(\Sigma)$ the σ–algebra induced from Σ. For fuzzy sets $\tilde{A}$, $\tilde{B} \in F(X)$, the sum $\tilde{A} \oplus \tilde{B}$, difference $\tilde{A} \ominus \tilde{B}$ and product $\tilde{A} \odot \tilde{B}$ are the fuzzy sets defined by, respectively

$$(\tilde{A} \oplus \tilde{B})(x) = min\{1, \tilde{A}(x) + \tilde{B}(x)\}$$
$$(\tilde{A} \ominus \tilde{B})(x) = max\{0, \tilde{A}(x) - \tilde{B}(x)\}$$
$$(\tilde{A} \odot \tilde{B})(x) = \tilde{A}(x) \cdot \tilde{B}(x)$$

Recall that a fuzzy measurable space (X, Σ) is said to be regular if Σ satisfies the condition:

$$\forall \alpha \in [0,1], \ \forall E \in \sigma(\Sigma) \ => \ \alpha \cdot \chi_E \in \Sigma$$

Where Σ is the σ–additive class of fuzzy sets [4,5].

More details about the the proerties of fuzzy measurable space and regular fuzzy measurable space can be found in [1,2,4,5].

Definition 1.1 Let H be a Hilbert space on **R**. H is said to be ordered, if a partial order $\leq$ on H is defined with the following conditions: for $\forall$ a, b, c, $d \in$ H

(1) $a \leqslant b,\ c \leqslant d \ => \ a+c \leqslant b+d$

(2) $\forall k \in \mathbf{R}^+,\ a \leqslant b \ => \ k \cdot a \leqslant k \cdot b$

(3) $\forall h \in \mathbf{H}^+,\ -h \leqslant a \leqslant h \ => \ \|a\| \leqslant \|b\|$

(4) $\mathbf{H}^+$ is complete.

Where $\mathbf{H}^+ = \{h \in H:\ \theta \leqslant h\}$, θ is the zero element of **H** and $\|h\| = \sqrt{<h, h>}$.

In this paper, we will always suppose that **H** is an ordered Hilbert space, and for distinction, we denote the limit operation in **H** by (H) lim.

2. Fuzzy Orthogonal Measures on Regular Fuzzy Measurable Space

Definition 2.1 Let (X, Σ) be a regular fuzzy measurable space and **H** an ordered Hilbert space. A mapping μ: $\Sigma \rightarrow \mathbf{H}^+$ is called an H–valued fuzzy orthogonal measure (abbr. HFOM) if the following conditions are satisfied:

$(H1)$ $\mu(\tilde{O}) = 0$ $(\tilde{O}(x) = 0,\ \forall x \in X)$;

$(H2)$ if $\tilde{A},\ \tilde{B} \in \Sigma$ and $\tilde{A} \bigcap \tilde{B} = \tilde{O}$ then $<\mu(\tilde{A}),\ \mu(\tilde{B})> = 0$;

$(H3)$ if $\tilde{A}_n \in \Sigma$ and $\sum_{n=1}^{\infty} \tilde{A}_n(x) \leqslant 1$ $(\forall x \in X)$ then $\mu(\bigoplus_{n=1}^{\infty} \tilde{A}_n) = \sum_{n=1}^{\infty} \mu(\tilde{A}_n)$.

The conditions (H2) and (H3) are called the orthogonality and σ–additivity of the HFOM μ, respectively.

Refer to [1, 2], it is easy to prove that an HFOM μ on the regular fuzzy measurable space (X, Σ) has the following properties:

Propostion 2.1 Let μ be an HFOM on regular fuzzy measurable space (X, Σ), then

(1) μ is finitely additive, i.e.

if $\tilde{A}_i \in \Sigma$ and $\sum_{i=1}^{n} \tilde{A}_i(x) \leqslant 1$ $(\forall x \in X)$ then $\mu(\bigoplus_{i=1}^{n} \tilde{A}_i) = \sum_{i=1}^{n} \mu(\tilde{A}_i)$

(2) μ is montone and subtractive, i.e.

if $\tilde{A},\ \tilde{B} \in \Sigma$ and $\tilde{A} \leqslant \tilde{B}$ then $\mu(\tilde{A}) \leqslant \mu(\tilde{B})$, $\mu(\tilde{B}) \ominus \tilde{A}) = \mu(\tilde{B}) + (-\mu(\tilde{A}))$

where $-\mu(\tilde{A})$ is the negative element of $\mu(\tilde{A})$ in **H**.

(3) μ is continuous, i.e.

if $\tilde{A}_n \in \Sigma$ and $\tilde{A}_n \uparrow \tilde{A}$ $(resp.\ \tilde{A}_n \downarrow \tilde{A})$ then $(H)\ \lim_{n\to\infty} \mu(\tilde{A}_n) = \mu(\tilde{A})$

Now we present the representation theorem of HFOM's, which demonstrates the structural characteristics of HFOM's:

Theorem 2.1 A mapping $\mu: \Sigma \to H^+$ is an HFOM on the regular fuzzy measurable space (X, Σ) if and only if there exixts a classical finite measure m: $\sigma(\Sigma) \to \mathbf{R}^+$ such that

$$<\mu(\tilde{A}),\ \mu(\tilde{B})> = \int_X (\tilde{A} \odot \tilde{B})(x)\, dm \qquad (\forall\ \tilde{A},\ \tilde{B} \in \Sigma)$$

where $\sigma(\Sigma)$ is is the σ-algebra induced from Σ.

Proof (1) Sufficiency: Suppose the conditions of theorem 2.1 hold. To prove that μ is an fuzzy orthogonal measure, we only need to prove that the condition (H3) holds. In fact, for $\tilde{A}_n \in \Sigma$ $(n = 1,2,3,...)$ and $\sum_{n=1}^{\infty} \tilde{A}_n(x) \leqslant 1$ $(\forall\, x \in X)$, we have

$$\|\mu(\bigoplus_{i=1}^{\infty} \tilde{A}_i) - \sum_{i=1}^{n} \mu(\tilde{A}_i)\|^2 = <\mu(\bigoplus_{i=1}^{\infty} \tilde{A}_i) - \sum_{i=1}^{n} \mu(\tilde{A}_i),\ \mu(\bigoplus_{i=1}^{\infty} \tilde{A}_i) - \sum_{i=1}^{n} \mu(\tilde{A}_i)>$$

$$= <\mu(\bigoplus_{i=1}^{\infty} \tilde{A}_i),\ \mu(\bigoplus_{i=1}^{\infty} \tilde{A}_i)> - 2\sum_{j=1}^{n} <\mu(\bigoplus_{i=1}^{\infty} \tilde{A}_i),\ \mu(\tilde{A}_j)> + \sum_{i=1}^{n}\sum_{j=1}^{n} <\mu(\tilde{A}_i),\ \mu(\tilde{A}_j)>$$

$$= \int_X [\left(\sum_{i=1}^{\infty} \tilde{A}_i(x)\right)^2 - 2\sum_{j=1}^{n}\left(\sum_{i=1}^{\infty} \tilde{A}_i(x) \cdot \tilde{A}_j(x)\right) + \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{A}_i(x) \cdot \tilde{A}_j(x)]\, dm$$

$$= \int_X [\sum_{i=1}^{\infty} \tilde{A}_i(x) - \sum_{i=1}^{n} \tilde{A}_i(x)]^2\, dm = \int_X [\sum_{i=n+1}^{\infty} \tilde{A}_i(x)]^2\, dm$$

Since $0 \leqslant \sum_{i=n+1}^{\infty} \tilde{A}_i(x) \leqslant 1$ and $\lim_{n\to\infty} \sum_{i=n+1}^{\infty} \tilde{A}_i(x) = 0$, Fatou's lemma [3] implies that

$$\lim_{n\to\infty} \|\mu(\bigoplus_{i=1}^{\infty} \tilde{A}_i) - \sum_{i=1}^{n} \mu(\tilde{A}_i)\|^2 = \lim_{n\to\infty} \int_X [\sum_{i=n+1}^{\infty} \tilde{A}_i(x)]^2\, dm = 0$$

i.e.

$$\mu(\bigoplus_{i=1}^{\infty} \tilde{A}_i) = (H)\lim_{n\to\infty} \sum_{i=1}^{n} \mu(\tilde{A}_i) = \sum_{i=1}^{\infty} \mu(\tilde{A}_i)$$

(2) Necessity: Suppose μ is a fuzzy orthogonal measure on (X, Σ). We define a non-negative set function m: $\sigma(\Sigma) \to \mathbf{R}^+$ by

$$m(E) = <\mu(\chi_E),\ \mu(\chi_E)> \qquad (\forall\ E \in \sigma(\Sigma))$$

Then we can prove that m is a classical finite measure on $(X, \sigma(\Sigma))$.

Since Σ is regular, from [5, theorem 2.5] we know that $\Sigma = F(\sigma(\Sigma)) = \{\tilde{A} \in F(X):\ \tilde{A}$ is $\sigma(\Sigma)$-measurable$\}$. Hence, for any $\tilde{A}$, $\tilde{B} \in \Sigma$, $\tilde{A}$ and $\tilde{B}$ are both $\sigma(\Sigma)$-measurable. From the properties of measurable functions [3], it is known that we can choose $k = k(n, m) \in N$, α_i, $\beta_i \in [0, 1]$ and disjoint sets $E_i \in \sigma(\Sigma)$ $(i = 1,2,...k)$ such that

$$\tilde{A}_n = \bigoplus_{i=1}^{k} \alpha_i \cdot \chi_{E_i}, \qquad \tilde{B}_m = \bigoplus_{i=1}^{k} \beta_i \cdot \chi_{E_i}$$

with $\tilde{A}_n \uparrow \tilde{A}$, $\tilde{B}_m \uparrow \tilde{B}$ and $k = k(n, m) \to \infty$ as $n, m \to \infty$.

From the regularity of Σ and the continuity of μ, we know $\tilde{A}_n$, $\tilde{B}_m \in \Sigma$ and

$$\mu(\tilde{A}_n) = \sum_{i=1}^{k} \alpha_i \cdot \mu(\chi_{E_i}), \qquad \mu(\tilde{B}_m) = \sum_{i=1}^{k} \beta_i \cdot \mu(\chi_{E_i})$$

Therefore

$$< \mu(\tilde{A}_n), \mu(\tilde{B}_m) > = \sum_{i=1}^{k} \sum_{j=1}^{k} \alpha_i \beta_j \cdot < \mu(\chi_{E_i}), \mu(\chi_{E_j}) >$$

$$= \sum_{i=1}^{k} \alpha_i \beta_i \cdot < \mu(\chi_{E_i}), \mu(\chi_{E_i}) > = \sum_{i=1}^{k} \alpha_i \beta_i \cdot m(E_i) = \int_X (\tilde{A}_n \odot \tilde{B}_m)(x)\, dm$$

Hence

$$< \mu(\tilde{A}), \mu(\tilde{B}) > = \lim_{n,m \to \infty} < \mu(\tilde{A}_n), \mu(\tilde{B}_m) >$$

$$= \lim_{n,m \to \infty} \int_X (\tilde{A}_n \odot \tilde{B}_m)(x)\, dm = \int_X (\tilde{A} \odot \tilde{B})(x)\, dm.$$

This ends the proof of theorem 2.1.

3. The Normal Family and Weak Convergence of HFOM's

Definition 3.1 Let (X, Σ) be a regular fuzzy measurable space and Φ a family of HFOM's on (X, Σ). Φ is said to be normal, if for any μ, $\nu \in \Phi$ and any $\tilde{A}$, $\tilde{B} \in \Sigma$, we have $< \mu(\tilde{A}), \nu(\tilde{B}) > = 0$ whenever $\tilde{A} \cap \tilde{B} = \tilde{O}$.

Propostion 3.1 Suppose regular fuzzy measurable space Φ is a family of HFOM's on the regular fuzzy measurable space (X, Σ). Then Φ is normal if and only if, for any μ, $\nu \in \Phi$, there exist s classical finite signed measure m on $(X, \sigma(\Sigma))$ such that

$$< \mu(\tilde{A}), \nu(\tilde{B}) > = \int_X (\tilde{A} \odot \tilde{B})(x)\, dm \qquad (\forall \tilde{A}, \tilde{B} \in \Sigma)$$

Definition 3.2 Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of HFOM's on the regular fuzzy measurable space (X, Σ). If for any $\tilde{A} \in \Sigma$, the sequence $\{\mu_n(\tilde{A})\}_{n=1}^{\infty}$ in H^+ is convergent to $\mu(\tilde{A})$, then we call μ: $\Sigma \to H^+$ the weak limit of $\{\mu_n\}_{n=1}^{\infty}$, or we say that $\{\mu_n\}_{n=1}^{\infty}$ converges weakly to μ.

We can prove that the weak limit have the following properties:

Proposition 3.2 Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of HFOM's on the regular fuzzy measurable space (X, Σ). If $\{\mu_n\}_{n=1}^{\infty}$ converges weakly to μ, then μ is also an HFOM on (X, Σ).

Proposition 3.3 Suppose Φ is a normal family of HFOM's on the regular fuzzy measurable space (X, Σ). Then

$$\Phi^* = \{\mu\text{: there exist } \{\mu_n\}_{n=1}^{\infty} \text{ in } \Phi \text{ such that } \{\mu_n\}_{n=1}^{\infty} \text{ converges weakly to } \mu\}$$

is also a normal family of FHOM's.

Theorem 3.1 Suppose Φ is a normal family of HFOM's on the regular fuzzy measurable space (X, Σ), and it forms a linear space. If Φ is closed under weak convergence, then the function $<\cdot, \cdot>_\Phi$: $\Phi\times\Phi\rightarrow \mathbf{R}$ defined by

$$<\mu, \nu>_\Phi = <\mu(\chi_X), \nu(\chi_X)>$$

is an inner product in Φ, and Φ is a Hilbert space. Moreover, $\{\mu_n\}_{n=1}^{\infty}$ converges strongly in Φ if and only if it converges weakly in Φ.

Proof It is easy to verify that $<\cdot, \cdot>_\Phi$ is an inner product in Φ. To prove that Φ is a Hilbert space, we introduce a norm $\|\cdot\|_\Phi$ by $\|\mu\|_\Phi=\sqrt{<\mu, \mu>_\Phi}$.

Suppose $\{\mu_n\}_{n=1}^{\infty}$ is a Cauchy sequence in Φ, i.e. $\|\mu_n-\mu_k\|_\Phi\rightarrow 0$ as n, k→∞. From Proposition 3.1 we know that, for $\forall \tilde{A}\in\Sigma$,

$$\|\mu_n(\tilde{A})-\mu_k(\tilde{A})\|^2 = <(\mu_n-\mu_k)(\tilde{A}), (\mu_n-\mu_k)(\tilde{A})> = \int_X \tilde{A}^2\, dm$$
$$\leqslant \int_X \chi_X\, dm = <(\mu_n-\mu_k)(\chi_X), (\mu_n-\mu_k)(\chi_X)>$$
$$= <\mu_n-\mu_k, \mu_n-\mu_k> = \|\mu_n-\mu_k\|_\Phi^2 \rightarrow 0 \quad (n, k\rightarrow\infty)$$

This implies that, for $\forall \tilde{A}\in\Sigma$, $\{\mu_n\}_{n=1}^{\infty}$ is a Cauchy sequence in H^+. Therefore, from the completeness of H^+, there exists $\mu(\tilde{A})\in H^+$ such that $(H)\ \lim_{n\rightarrow\infty}\mu_n(\tilde{A})=\mu(\tilde{A})$ ($\forall \tilde{A}\in\Sigma$), i.e. $\mu_n => \mu$. Since Φ is closed under weak convergence, we have $\mu\in\Phi$. So Φ is complete, and consequently, Φ is a Hilbert space.

Further, suppose $\{\mu_n\}_{n=1}^{\infty}$ converges to μ strongly in Φ, then for $\forall \tilde{A}\in\Sigma$, we have

$$\|\mu_n(\tilde{A})-\mu(\tilde{A})\|^2 = <(\mu_n-\mu)(\tilde{A}), (\mu_n-\mu)(\tilde{A})> = \int_X \tilde{A}^2\, dm$$
$$\leqslant \int_X \chi_X\, dm = <(\mu_n-\mu)(\chi_X), (\mu_n-\mu)(\chi_X)>$$
$$= <\mu_n-\mu, \mu_n-\mu> = \|\mu_n-\mu\|_\Phi^2 \rightarrow 0 \quad (n,\rightarrow\infty)$$

That is, $\{\mu_n\}_{n=1}^{\infty}$ converges to μ weakly in Φ.

On the other hand, if $\{\mu_n\}_{n=1}^{\infty}$ converges to μ weakly in Φ, i.e. $(H)\ \lim_{n\to\infty}\mu_n(\tilde{A}) = \mu(\tilde{A})$ holds for $\forall\ \tilde{A}\in\Sigma$. Taking $\tilde{A} = \chi_X$, we get

$$\|\mu_n - \mu\|_{\Phi}^2 = <\mu_n - \mu,\ \mu_n - \mu> = <(\mu_n - \mu)(\chi_X),\ (\mu_n - \mu)(\chi_X)>$$
$$= \|\mu_n(\chi_X) - \mu(\chi_X)\|^2 \to 0 \quad (n\to\infty).$$

This means that $\{\mu_n\}_{n=1}^{\infty}$ converges to μ strongly in Φ. Hence the stronge convergence and weak convergence are equivalent in Φ.

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