On the convergence of the fuzzy valued functional

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Abstract: In this paper, we define a fuzzy valued functional on fuzzy measure spaces, discuss its important properties and give its convergence theorem.

keywords: measurable fuzzy valued functions; integrable fuzzy valued functions; fuzzy valued functional.

1 Introduction

It was Sugeno [2] who defined the first functionals (a fuzzy integral) for the extension of fuzzy measure, subsequently various functionals have been proposed to deal with particular types of fuzzy measure(klement [3], weber [4], Matloka [5] ect.). But most functionals are defined on the spaces of real measurable functions. Nevertheless, fuzzy valued functionals are seldom studied. This paper will be motivated by the work of M.J.Bolanos etc [1] who investigated convergence properties of the montone expectation. First we define the fuzzy valued functional. And then we discuss its properties and give its convergence theorem.

2 Basis operations and definitions

In the paper, the following notations will be used.

R^+ denotes $[0,\infty)$. X is an arbitrary fixed set. $\mathscr{A}$ is a σ – algebra formed by the subsets of X. (X, $\mathscr{A}$) is a measurable space. Let $I_{R^+} = \{\bar{a} = [a^-, a^+] : a^- \leqslant a^+, a^-, a^+ \in R^+\}$

Then the elements in the set I_{R^+} are called interval numbers. on the interval numbers set, we make following definitions:

For every $\bar{a}, \bar{b} \in I_{R^+}$. $\bar{a} + \bar{b} = [a^- + b^-, a^+ + b^+]$; $\quad \bar{a} \cdot \bar{b} = [a^- \cdot b^-, a^+ \cdot b^+]$;

$k\bar{a} = [ka^-, ka^+]$ where $k \geqslant 0$; $\bar{a} \leqslant \bar{b}$ *iff* $a^- \leqslant b^-, a^+ \leqslant b^+$.

Especially $\bar{a} = \bar{b}$ *iff* $a^- = b^-$, $a^+ = b^+$.

Let $d(\bar{a}, \bar{b}) = max\{|a^- - b^-|, |a^+ - b^+|\}$.

Then (I_{R^+}, d) is an Hausdorff metric space.

For a sequence of interval numbers $\{\bar{a}_n\}$. We say that $\bar{a}_n \to \bar{a}$, if $d(\bar{a}_n, \bar{a}) \to 0$ $(n \to \infty)$.

Obviously, $\bar{a}_n \to \bar{a}$ iff $a_n^- \to a^-, a_n^+ \to a^+ (n \to \infty)$

Definition 2.1 Let A: $R^+ \to [0,1]$. If there exists an $x_0 \in R^+$, such that $A(x_0) = 1$ and for any $\lambda \in (0,1]$, the level set $A_\lambda = \{x \in R^+ : A(x) \geqslant \lambda\} \in I_{R^+}$.

Then we call A a fuzzy number on R^+ . We denote all fuzzy numbers on R^+ as $F(R^+)$.

Let $A, B \in F(R^+)$, for any $\lambda \in (0,1], k \geqslant 0$. we define:

$(A+B)_\lambda = A_\lambda + B_\lambda$; $(kA)_\lambda = kA_\lambda$; $A \leqslant B$ iff $A_\lambda \leqslant B_\lambda$

$$a(x) = \begin{cases} 1 & x = a \\ 0 & x \neq a \end{cases} \quad \text{where } a \in R^+ .$$

Obviously, $a_\lambda = [a, a] = \{a\}$, and $a \in F(R^+)$.

Let $\bar{d}(A,B) = \sup_{0<\lambda\leqslant 1} d(A_\lambda, B_\lambda)$. Then $(F(R^+), \leqslant)$ constitutes a partial ordered set,

And $\bar{d}$ is a metric on $F(R^+)$.

For a sequence of fuzzy numbers $\{A_n\} \subset F(R^+)$, $A \in F(R^+)$. We say that $\{A_n\}$ is convergent to A iff $d((A_n)_\lambda, A_\lambda) \to 0$ $(n \to \infty)$. For any $\lambda \in (0,1]$. Simply write as $\lim_{n\to\infty} A_n = A$ or $A_n \to A$. Evidently. $A_n \to A$ iff for any $\lambda \in (0,1]$, $(A_n)_\lambda^- \to A_\lambda^-, (A_n)_\lambda^+ \to A_\lambda^+$ $(n \to \infty)$.

Definition 2.2 Let $\mu: \mathscr{A} \to [0,1]$ be a mapping. If the following conditions are satisfied:

(1) $\mu(\emptyset) = 0, \mu(X) = 1$

(2) $A, B \in \mathscr{A}$, $A \subset B$ implies $\mu(A) \leqslant \mu(B)$.

(3) If $\{A_n\}$ is a monotone sequence of elements in $\mathscr{A}$. Then $\lim_{n\to\infty}\mu(A_n) = \mu(\lim_{n\to\infty} A_n)$,

We call μ a fuzzy measure; $(X, \mathscr{A}, \mu)$ is said to a fuzzy measure space.

Let $\tilde{f}: X \to F(R^+)$. we say that $\tilde{f}$ is a fuzzy valued function.

For arbitrary $\lambda \in (0,1]$. we define $(\tilde{f}(x))_\lambda = f_\lambda(x) = [f_\lambda^-(x), f_\lambda^+(x)] \in I_R^+$.

Definition 2.3 Let $\tilde{f}$ be a fuzzy valued function with respect to $(X, \mathscr{A}, \mu)$. For any $\lambda \in (0,1]$ and $t \geqslant 0$, $\tilde{f}$ is called a measurable fuzzy valued function, if $(f_\lambda^-)_t = \{x \in X: f_\lambda^-(x) \geqslant t\} \in \mathscr{A}$ and $(f_\lambda^+)_t = \{x \in X: f_\lambda^+(x) \geqslant t\} \in \mathscr{A}$.

Definition 2.4 Let $\tilde{f}$ be a measurable fuzzy valued function with respect to $(X, \mathscr{A}, \mu)$. If for any $\lambda \in (0,1]$, its Lebesgue integral $\int_0^{+\infty} \mu(f_\lambda^-)_t < +\infty$ and $\int_0^{+\infty} \mu(f_\lambda^+)_t < +\infty$.

Then $\tilde{f}$ is called integrable with respect to μ on $(X, \mathscr{A}, \mu)$. It is simply called $\mu-$ integrable.

Lemma 2.1 Let $H: (0,1] \to I_{R^+}, \lambda \to H(\lambda) = [m_\lambda, n_\lambda]$ and with the property:

$\lambda_1 < \lambda_2$ *implies* $[m_{\lambda_2}, n_{\lambda_2}] \subset [m_{\lambda_1}, n_{\lambda_1}]$. where $\lambda_1, \lambda_2 \in (0,1]$. Define $A = \bigcup_{\lambda \in (0,1]} \lambda H(\lambda)$.

Then $A \in F(R^+)$ and $A_\lambda = \bigcap_{n=1}^{\infty} H(\lambda_n)$ where $\lambda_n = (1 - \frac{1}{n+1})\lambda$ $(\lambda > 0)$.

The proof refer to [6]

3 the properties of the fuzzy valued functional

In this section, we first define the fuzzy valued functional on $(X, \mathscr{A}, \mu)$, and then we mainly discuss the general properties of this kind functional.

Definition 3.1 Let $\tilde{f}$ be a μ - integrable fuzzy valued function on (X , $\mathscr{A}$, μ).

We define $E_\mu(\tilde{f}) = \bigcup_{\lambda\in(0,1]} \lambda[\int_0^{+\infty}\mu(f_\lambda^-)_t dt, \int_0^{+\infty}\mu(f_\lambda^+)_t dt]$.

Then $E_\mu(\tilde{f})$ is called the fuzzy valued functional with respect to (X , $\mathscr{A}$, μ). Above integrals are Lebesgue integrals.

For arbitrary $\lambda \in (0,1]$ and $t \geqslant 0$, $(f_\lambda^-)_t \subset (f_\lambda^+)_t$ alwadys holds.

Consequently, $\mu(f_\lambda^-)_t \leqslant \mu(f_\lambda^+)_t$ and $\int_0^{+\infty}\mu(f_\lambda^-)_t dt \leqslant \int_0^{+\infty}\mu(f_\lambda^+)_t dt < +\infty$.

Hence $E_\mu(\tilde{f})$ is always significant.

Theorem 3.1 If $\tilde{f}$ is a μ - integrable fuzzy valued function on (X , $\mathscr{A}$, μ).

Then $E_\mu(\tilde{f}) \in F(R^+)$ and $(E_\mu(\tilde{f}))_\lambda = \bigcap_{n=1}^{\infty}[\int_0^{+\infty}\mu(f_{\lambda_n}^-)_t dt, \int_0^{+\infty}\mu(f_{\lambda_n}^+)_t dt]$,

where $\lambda_n = (1 - \frac{1}{n+1})\lambda$ and $\lambda > 0$

proof Let $H:(0,1] \to I_{R^+}$

$$\lambda \to H(\lambda) = [\int_0^{+\infty}\mu(f_\lambda^-)_t dt, \int_0^{+\infty}\mu(f_\lambda^+)_t dt]$$

Evidently, for arbitrary $x \in X$, We have $(\tilde{f}(x))_{\lambda_2} \subset (\tilde{f}(x))_{\lambda_1}$ whenever $\lambda_1 < \lambda_2$

i.e., $[f_{\lambda_2}^-(x), f_{\lambda_2}^+(x)] \subset [f_{\lambda_1}^-(x), f_{\lambda_1}^+(x)]$.

Therefore, by lemma 2.1, the proof is completed.

property 1. **Let $\tilde{f}$ and $\tilde{g}$ be two μ - integrable fuzzy valued functions on (X , $\mathscr{A}$, μ).**

If **$\tilde{f}(x) \leqslant \tilde{g}(x)$ for all $x \in X$. Then $E_\mu(\tilde{f}) \leqslant E_\mu(\tilde{g})$.**

proof For any $\lambda \in (0,1]$ and $t \geqslant 0$. obviously we have $(f_\lambda^-)_t \subset (g_\lambda^-)_t, (f_\lambda^+)_t \subset (g_\lambda^+)_t$.

Consequently $\mu(f_\lambda^-)_t \leqslant \mu(g_\lambda^-)_t$, $\mu(f_\lambda^+)_t \leqslant \mu(g_\lambda^+)_t$.

And $[\int_0^{+\infty}\mu(f_\lambda^-)_t dt, \int_0^{+\infty}\mu(f_\lambda^+)_t dt] \leqslant [\int_0^{+\infty}\mu(g_\lambda^-)_t dt, \int_0^{+\infty}\mu(g_\lambda^+)_t dt]$.

Thus $E_\mu(\tilde{f}) \leqslant E_\mu(\tilde{g})$.

property 2 Let μ_1 and μ_2 be fuzzy measure on (X , $\mathscr{A}$). $\tilde{f}$ is integrable with respect to μ_1 and μ_2 . If $\mu_1(A) \leqslant \mu_2(A)$ where any $A \in \mathscr{A}$. Then $E_{\mu_1}(\tilde{f}) \leqslant E_{\mu_2}(\tilde{f})$.

proof Since $\tilde{f}$ is integrable, it is certain measurable.

So for every $\lambda \in (0,1]$ and $t \geqslant 0$, we get $(f_\lambda^-)_t \in \mathscr{A}$, $(f_\lambda^+)_t \in \mathscr{A}$.

Consequently $\mu_1(f_\lambda^-)_t \leqslant \mu_2(f_\lambda^-)_t$, *and* $\mu_1(f_\lambda^+)_t \leqslant \mu_2(f_\lambda^+)_t$.

We obtain $[\int_0^{+\infty}\mu_1(f_\lambda^-)_t dt, \int_0^{+\infty}\mu_1(f_\lambda^+)_t dt] \leqslant [\int_0^{+\infty}\mu_2(f_\lambda^-)_t dt, \int_0^{+\infty}\mu_2(f_\lambda^+)_t dt]$.

Hence $E_{\mu_1}(\tilde{f}) \leqslant E_{\mu_2}(\tilde{f})$

property 3 If $\tilde{f}$ is a constant fuzzy valued function. i.e., for any $x \in X$, $\tilde{f}(x) = A \in F(R^+)$. Then $E_\mu(\tilde{f}) = A$

proof First, for any $\lambda \in (0,1]$ *and* $x \in X$.

We have $f_\lambda^-(x) = A_\lambda^- \in R^+, f_\lambda^+(x) = A_\lambda^+ \in R^+$.

Thus, $(f_\lambda^-)_t = \{x \in X : f_\lambda^-(x) \geqslant t\} = X.$ whenever $0 \leqslant t \leqslant A_\lambda^-$;

$(f_\lambda^-)_t = \{x \in X : f_\lambda^-(x) \geqslant t\} = \emptyset$ whenever $t > A_\lambda^-$.

Consequently, $\int_0^{+\infty} \mu(f_\lambda^-)_t dt = \int_0^{A_\lambda^-} \mu(X) dt + \int_{A_\lambda^-}^{+\infty} \mu(\emptyset) dt = A_\lambda^-$

Similarly we can obtain $\int_0^{+\infty} \mu(f_\lambda^+)_t dt = A_\lambda^+$.

Therefore, $[\int_0^{+\infty} \mu(f_\lambda^-)_t dt, \int_0^{+\infty} \mu(f_\lambda^+)_t dt] = [A_\lambda^-, A_\lambda^+] = A_\lambda$.

It follows that $E_\mu(\tilde{f}) = \bigcup_{\lambda \in (0,1]} \lambda A_\lambda = A$

property 4 Let $\tilde{f}$ and $a + b\tilde{f}$ be $\mu-$integrable fuzzy functions on $(X, \mathscr{A}, \mu)$. where $a \geqslant 0, b > 0$. Then we have $E_\mu(a + b\tilde{f}) = a + bE_\mu(\tilde{f})$.

proof For any $\lambda \in (0,1]$. By the definitions of fuzzy numbers and their level sets, we know that $(a + b\tilde{f})_\lambda = [a + bf_\lambda^-, a + bf_\lambda^+]$ holds.

Hence, $(a + bf_\lambda^-)_t = \{x \in X : a + bf_\lambda^-(x) \geqslant t\} = X$ whenever $0 \leqslant t \leqslant a$.

Let $\frac{t-a}{b} = y$. we obtain $\int_a^{+\infty} \mu(a + bf_\lambda^-)_t dt = \int_a^{+\infty} \mu\{x \in X : f_\lambda^-(x) \geqslant \frac{t-a}{b}\} dt$

$$= b\int_0^{+\infty} \mu(f_\lambda^-)_y dy = b\int_0^{+\infty} \mu(f_\lambda^-)_t dt \quad \text{whenever} \quad t > a.$$

Thus $\int_0^{+\infty} \mu(a + bf_\lambda^-)_t dt = \int_0^a \mu(X) dt + \int_a^{+\infty} \mu(a + bf_\lambda^-)_t dt = a + b\int_0^{+\infty} \mu(f_\lambda^-)_t dt$.

Similarly, we can get $\int_0^{+\infty} \mu(a + bf_\lambda^+) dt = a + b\int_0^{+\infty} \mu(f_\lambda^+)_t dt$

This follow that $E_\mu(a + b\tilde{f}) = \bigcup_{\lambda \in (0,1]} \lambda\{[a,a] + b[\int_0^{+\infty} \mu(f_\lambda^-)_t dt, \int_0^{+\infty} \mu(f_\lambda^+)_t dt]\}$

$$= a + bE_\mu(\tilde{f}).$$

4 The convergence properties of the fuzzy valued functional

In this section, we describe some important results of this kind of fuzzy valued functional. First, the validity of Levi's monotone convergence theorems and Fatou's lemma is proved. Next, we obtain results similar to Lebesgue's dominated convergence theorem.

Throughout this section, we always discuss the problems on fuzzy measure space $(X, \mathscr{A}, \mu)$.

Theorem 4.1 Let $\tilde{f}_n: X \to F(R^+)$ be a monotone sequence of μ − integrable fuzzy valued functions which converges to $\tilde{f}: X \to F(R^+)$.

Then $\tilde{f}$ is μ − integrable too, and $\lim_{n\to\infty} E_\mu(\tilde{f}_n) = E_\mu(\tilde{f})$.

proof Without loss of generality, we assume that $\{\tilde{f}_n\}$ is monotone decreasing.

Then for every $n \in N$ and $\lambda \in (0, 1]$. we have $\tilde{f}(x) \leqslant \tilde{f}_n(x)$. $\quad x \in X$

Consequently, we obtain $\int_0^{+\infty} \mu(f_\lambda^-)_t dt \leqslant \int_0^{+\infty} \mu(f_{n\lambda}^-)_t dt < +\infty$ and

$$\int_0^{+\infty} \mu(f_\lambda^+)_t dt \leqslant \int_0^{+\infty} \mu(f_{n\lambda}^+)_t dt < +\infty$$

$$\text{where} \quad (\tilde{f}_n(x))_\lambda = [f_{n\lambda}^-(x), f_{n\lambda}^+(x)], x \in X$$

Therefore, $\tilde{f}$ is μ − integrable.

Let $(f_{n\lambda}^-)_t = \{x \in X: f_{n\lambda}^-(x) \geqslant t\}$,

$(f_{n\lambda}^+)_t = \{x \in X: f_{n\lambda}^+(x) \geqslant t\}$ where $\lambda \in (0, 1]$ and $t \geqslant 0$

Obviously, for every $n \in N$ $\{(f_{n\lambda}^-)_t\}$ is a monotone decreasing sequence of sets in $\mathscr{A}$

And $\lim_{n\to\infty}(f_{n\lambda}^-)_t = \bigcap_{n=1}^{\infty}(f_{n\lambda}^-)_t = (f_\lambda^-)_t$.

In fact, if $x \in \bigcap_{n=1}^{\infty}(f_{n\lambda}^-)_t$. Then for every $n \in N$, we have $x \in (f_{n\lambda}^-)_t$ i.e. $f_{n\lambda}^-(x) \geqslant t$.

As $\tilde{f}_n \to \tilde{f}$, we get $\lim_{n\to\infty} f_{n\lambda}^-(x) = f_\lambda^-(x) \geqslant t$. i.e, $x \in (f_\lambda^-)_t$

Conversely, if $x \notin \bigcap_{n=1}^{\infty}(f_{n\lambda}^-)_t$. Then there exists at least a $n_0 \in N$, such that $x \notin (f_{n_0\lambda}^-)_t$

i.e. $f_{n_0\lambda}^-(x) < t$. since $\{\tilde{f}_n\}$ is a decreasing sequence.

Consequently, $f_\lambda^-(x) \leqslant f_{n_0\lambda}^-(x) < t$. $i.e., x \notin (f_\lambda^-)_t$.

Hence, $\bigcap_{n=1}^{\infty}(f_{n\lambda}^-)_t = (f_\lambda^-)_t$

By monotone convergence theorem of Lebesgue integrals and continuity of fuzzy measure μ,

We obtain $\lim_{n\to\infty}\int_0^{+\infty} \mu(f_{n\lambda}^+)_t dt = \int_0^{+\infty} \mu(\lim_{n\to\infty}(f_{n\lambda}^-)_t) dt = \int_0^{+\infty} \mu(f_\lambda^-)_t dt)$.

Similarly, we have $\lim_{n\to\infty}\int_0^{+\infty} \mu(f_{n\lambda}^+)_t dt = \int^{+\infty} \mu(f_\lambda^+)_t dt$

Therefore, by convergent definitions of the sequence of interval numbers,

We have $\lim_{n\to\infty} E_\mu(\tilde{f}_n) = E_\mu(\tilde{f})$.

Theorem 4.2 If $\tilde{f}$ and $\tilde{g}$ are measurable fuzzy valued function on with $\tilde{f}(x) \leqslant \tilde{g}(x)$

For any $x \in X$, and $\tilde{g}$ is μ − integrable. Then $\tilde{f}$ is μ − integrable too.

proof obvious

Theorem 4.3 Let $\{\tilde{f}_n\}$ be a sequence of μ − integrable fuzzy functions on X, If $\tilde{f}_n(x) \leqslant \tilde{g}(x)$ for each $n \in N$ and $\tilde{g}$ is μ − integrable. Then $\limsup_{n\to\infty} E_\mu(\tilde{f}_n) \leqslant E_\mu(\limsup_{n\to\infty} \tilde{f}_n)$.

proof For each $x \in X$. Let $\tilde{p}_n(x) = \sup_{i \geq n} \tilde{f}_i(x)$ evidently, $\{\tilde{p}_n\}$ is a decreasing sequence of functions. $\lim_{n \to \infty} \tilde{p}_n(x) = \limsup_{n \to \infty} \tilde{f}_n(x)$ and $\tilde{p}_n(x) \leqslant \tilde{g}(x)$.

By theorem 4.2, we know that $\tilde{p}_n$ is μ - integrable

From theorem 4.1, we have $\lim_{n \to \infty} E_\mu(\tilde{p}_n) = E_\mu(\limsup_{n \to \infty} \tilde{f}_n)$.

Since, $\tilde{f}_n(x) \leqslant \tilde{p}_n(x)$ for any $n \in N$ and $x \in X$.

By property1, we can obtain $E_\mu(\tilde{f}_n) \leqslant E_\mu(\tilde{p}_n)$

Therefore, $\limsup_{n \to \infty} E_\mu(\tilde{f}_n) \leqslant \limsup_{n \to \infty} E_\mu(\tilde{p}_n) = \lim_{n \to \infty} E_\mu(\tilde{p}_n) = E_\mu(\limsup_{n \to \infty} \tilde{f}_n)$

Corollary 4.1 If $\tilde{f}_n$ is a sequence of μ - integrable fuzzy valued functions on X

Then $E_\mu(\liminf_{n \to \infty} \tilde{f}_n) \leqslant \liminf_{n \to \infty} E_\mu(\tilde{f}_n)$

Theorem 4.4 Let $\tilde{f}_n$ be a sequence of measurable fuzzy valued functions and $\tilde{f}_n \to \tilde{f}$. If there exists a μ - integrable fuzzy valued function $\tilde{g}$ such that $\tilde{f}_n(x) \leqslant \tilde{g}(x)$ for each $n \in N$ and all $x \in X$

Then $\tilde{f}$ and $\tilde{f}_n$ are μ - integrable and $\lim_{n \to \infty} E_\mu(\tilde{f}_n) = E_\mu(\tilde{f})$.

proof obviously for any $n \in N$, $\tilde{f}$ and $\tilde{f}_n$ are μ - integrable.

By corollary 4.1 and theorem 4.3 we have

$E_\mu(\liminf_{n \to \infty} \tilde{f}_n) \leqslant \liminf_{n \to \infty} E_\mu(\tilde{f}_n) \leqslant \limsup_{n \to \infty} E_\mu(\tilde{f}_n) \leqslant E_\mu(\limsup_{n \to \infty} \tilde{f}_n)$.

Since $\lim_{n \to \infty} \tilde{f}_n = \tilde{f}$. It follows that $\liminf_{n \to \infty} \tilde{f}_n = \limsup_{n \to \infty} \tilde{f}_n = \tilde{f}$.

Consequently, $\liminf_{n \to \infty} E_\mu(\tilde{f}_n) = \limsup_{n \to \infty} E_\mu(\tilde{f}_n) = E_\mu(\tilde{f})$ .i.e., $\lim_{n \to \infty} E_\mu(\tilde{f}_n) = E_\mu(\tilde{f})$.

References

[1] M.J.Bolanos etc.. Convergence properties of the monotone expectation and its application to the extension of fuzzy measures. Fuzzy sets and Systems - (1989)201 - 212.

[2] M.Sugeno, Theory of fuzzy integrals and its applications, Thesis, Tokyo Institute of Technology (1974).

[3] E.P.Klement, Integration of fuzzy - valued functions, Rev. Roumaine Math. pures Appl. 30(1985)375 - 384.

[4] S.Weber, Two integrals and some modified versions: critical remarks, Fuzzy Sets and Systems 20(1986) 97 - 105.

[5] M.Matloka, on fuzzy integral, proc.polish Symp Interval and Fuzzy Mathematics -7(1986)163 - 170.

[6] Cheng zhong Luo and De me Wang, Extension of the integral of interval - valued functions and integrals of fuzzy - valued functions, fuzzy Math.3(1983)45 - 52(in Chinese).