Theory of Convergence of L-nets in Topological Molecular Lattices

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Abstract: In this paper, we introduce the concepts of L-nets and its convergence in topological molecular lattices[4], which is a generalization of the papers[1,4,5], and systematically discuss their properties and the characterizations of continuous generalized order-homomorphism by means of convergence the theory of L-net.

Keywords: L-net; limit point; cluster point; continuous order-homomorphism

1. Preliminaries

Throughout this paper, L and L_1 will denote completely distributive lattices, M and M_1 will denote the set of all moleculae in L and L_1 respectively, while $(L(M),\eta)$ and $(L_1(M_1),\eta_1)$ denote topological molecular lattices[4], the elements of η or η_1 will be called closed elements. Since there is no 'pseudo—complement', open and closed element are not dual concepts. A^- will denote the closure of $A\in L$, $\eta(e)=\{P\in\eta: e\nleq P\}$ and the elements in $\eta(e)$ are said to be R-neighborhood of $e\in M$.

2. Convergence of L-nets

In this section, we introduce the notions of limit points, cluster points and convergence of L-net, systematically discuss various properties of them, and so establish the convergence theory of L-nets.

Definition 2. 1. Let $(D,\leqslant)$ be a directed set. Then the mapping $S: D \longrightarrow L$ is called L-net in L. For each $n\in D$, put $S(n)=A_n$, then the net S will be denoted by $\{A_n: n\in D\}$.

Definition 2. 2. Let $\{A_n: n\in D\}$ be an L-net in $(L(M),\eta)$ and $e\in M$.

(1) e is called a limit point of $\{A_n: n\in D\}$ if for each $P\in\eta(e)$, there is an $m\in D$ such that $A_n\nleq P$ for all $n\geqslant m$.

(2) e is called a cluster point of $\{A_n : n\in D\}$ if for each $P\in\eta(e)$ and each $n\in D$, there is an $m\in D$ such that $m\geqslant n$ and $A_m\not\leqslant P$.

(3) $\underline{\lim}A_n$ is the union of all limit points of $\{A_n : n\in D\}$.

(4) $\overline{\lim}A_n$ is the union of all cluster points of $\{A_n : n\in D\}$.

(5) If $\underline{\lim}A_n=\overline{\lim}A_n=A$, then we say that A is the limit of $\{A_n : n\in D\}$ or say that $\{A_n : n\in D\}$ converges to A, in symbol $\lim A_n=A$.

From Definition 2. 2 we have

Theorem2. 1. *Let $\{A_n : n\in D\}$ be an L-net in $(L(M),\eta)$, then*

$$\underline{\lim}A_n\leqslant\overline{\lim}A_n$$

Theorem2. 2. *Let $\{A_n : n\in D\}$ be an L-net in $(L(M),\eta)$, and $e\in M$.*

(1) *$e\leqslant\underline{\lim}A_n$ iff e is a limit point of $\{A_n : n\in D\}$*

(2) *$e\leqslant\overline{\lim}A_n$ iff e is a cluster point of $\{A_n : n\in D\}$*

Proof. (1) In case $e\leqslant\underline{\lim}A_n$ and $P\in\eta(e)$. Since $e\not\leqslant P$ implies $\underline{\lim}A_n\not\leqslant P$, we have a limit point b of $\{A_n : n\in D\}$ with $b\not\leqslant P$ i. e. $P\in\eta(b)$, therefore there exists an $m\in D$ such that $A_n\not\leqslant P$ for all $n\geqslant m$. This shows that e is a limit point of $\{A_n : n\in D\}$. Conversely, if e is a limit point of $\{A_n : n\in D\}$, then $e\leqslant\underline{\lim}A_n$ by Defintion2. 2.

(2) The proof is similar to that of (1).

Corollary2. 3. *Let $\{A_n : n\in D\}$ be an L-net in $(L(M),\eta)$, then $\lim A_n=A$ iff the followings hold:*

(1) *If $e\leqslant A$, then e is a limit point of $\{A_n : n\in D\}$;*

(2) *If e is a cluster point of $\{A_n : n\in D\}$, then $e\leqslant A$.*

Theorem2. 4. *Let $\{A_n : n\in D\}$ is an L-net in $(L(M),\eta)$, then* $\overline{\lim}A_n=\bigwedge_{n\in D}(\bigvee_{m\geqslant n}A_m)^-$

Proof. From Definition2. 2 and Theorem2. 2 we have $e\leqslant\overline{\lim}A_n$ iff for each $P\in\eta(e)$ and each $n\in D$, there is an $m\in D$ such that $m\geqslant n$ and $A_m\not\leqslant P$ iff for each $P\in\eta(e)$ and each $n\in D$ there exists an $m\in D$ such that $\bigvee_{m\geqslant n}A_m\not\leqslant P$ iff for each $n\in D$, $e\leqslant(\bigvee_{m\geqslant n}A_m)^-$ iff $e\leqslant\bigwedge_{n\in D}(\bigvee_{m\geqslant n}A_m)^-$, where $e\in M$.

Theorem2. 5. *Let $\{A_n : n\in D\}$ be an L-net in $(L(M),\eta)$, put $\Omega=\{H : H$ is an arbitrary cofinal subset of $D\}$. Then*

$$\underline{lim}A_n = \bigwedge_{H\in\Omega}(\bigvee_{m\in H} A_m)^-$$

Proof. If $e \leqslant \underline{lim}A_n$ and $P\in\eta(e)$, then we have $n_0\in D$ satisfying $A_n \not\leqslant P$ for all $n\geqslant n_0$. Arbitrarily choose a cofinal subset H of D, then there exists an $m\in H$ such that $m\geqslant n_0$ and $A_m\not\leqslant P$. This implies that $e\leqslant(\bigvee_{m\in H} A_m)^-$ and so $\underline{lim}A_n \leqslant \bigwedge_{H\in\Omega}(\bigvee_{m\in H} A_m)^-$.

Conversely, now suppose e is not in $\underline{lim}A_n$, i. e. $e\not\leqslant\underline{lim}A_n$. then there is a $P\in\eta(e)$ such that for all n in D we can choose $m(n)\geqslant n$ with $A_m(n)\leqslant P$. Let $H_p=\{m(n):n\in D\}$. It is clear that H_p is a cofinal subset of D for which $\bigvee_{m(n)\in H_p} A_{m(n)}\leqslant P$. Hence $e\not\leqslant(\bigvee_{m(n)\in H_p} A_{m(n)})^-$ and hence $\bigwedge_{H\in\Omega}(\bigvee_{m\in H} A_m)^-\leqslant\underline{lim}A_n$.

Corollary2. 6. *Let $\{A_n:n\in D\}$ is an L-net in $(L(M),\eta)$, then*

(1) *$\underline{lim}A_n$ and $\overline{lim}A_n$ are closed elements in $(L(M),\eta)$.*

(2) *$\underline{lim}A_n=\underline{lim}(A_n)^-$.*

(3) *$\overline{lim}A_n=\overline{lim}(A_n)^-$.*

(4) *If $A_n=A$ for all $n\in D$, then $limA_n=A$.*

Now we discuss some relationships between two L-nets in $(L(M),\eta)$. From Definition2. 1 we have

Theorem2. 7. *Let $\{A_n:n\in D\}$ and $\{B_n:n\in D\}$ be two L-nets $(L(M),\eta)$, then*

(1) *If $A_n\leqslant B_n$ for every $n\in D$, then $\underline{lim}A_n\leqslant\underline{lim}B_n$, $\overline{lim}A_n\leqslant\overline{lim}B_n$*

(2) *$\overline{lim}(A_n\vee B_n)=\overline{lim}A_n\vee\overline{lim}B_n$* (3) *$\underline{lim}(A_n\vee B_n)\geqslant\underline{lim}A_n\vee\underline{lim}B_n$*

(4) *$\overline{lim}(A_n\wedge B_n)\leqslant\overline{lim}A_n\wedge\overline{lim}B_n$* (5) *$\underline{lim}(A_n\wedge B_n)\leqslant\underline{lim}A_n\wedge\underline{lim}B_n$*

(6) *If $\{B_n:n\in D\}$ is convergent, then $\overline{lim}(A_n\vee B_n)=\overline{lim}A_n\vee limB_n$ and $\underline{lim}(A_n\vee B_n)=\underline{lim}A_n\vee limB_n$*

(7) *If $\{A_n:n\in D\}$ and $\{B_n:n\in D\}$ are both convergent, then so is $\{A_n\vee B_n:n\in D\}$, and $lim(A_n\vee B_n)=limA_n\vee limB_n$*

(8) *If $\{A_n\vee B_n:n\in D\}$ is convergent, and if $\overline{lim}A_n\wedge B\,\overline{lim}_n=0$, then $\{A_n:n\in D\}$ and $\{B_n:n\in D\}$ are both convergent.*

Next we disucuss relationships between an L-net and its subnets.

Definition2. 3. Let $\triangle=\{\triangle(n):n\in D\}$ and $T=\{T(m):m\in E\}$ be two L-nets in $(L(M),\eta)$. T is called a subnet of $\triangle$ if thene exists a mapping $N:E\longrightarrow D$ such that (1) $T=\triangle oN$, (2) for each $n\in D$ there is an $m\in E$ with $N(k)\geqslant n$ whenever $k\geqslant m$ ($k\in E$).

Theorem2. 8. *Let* $\{B_m : m\in E\}$ *be a subnet of* $\{A_n : n\in D\}$, *then*

(1) $\underline{\lim} A_n \leqslant \underline{\lim} B_m$

(2) $\overline{\lim} B_m \leqslant \overline{\lim} A_n$

(3) *If* $\{A_n : n\in D\}$ *converges to* A, *then every subnet also converges to* A.

The theorem follows directly from Definition2. 2 and Definition2. 3.

Theorem2. 9. *Let* $\{A_n : n\in D\}$ *be an L-net and* $G=\{T : T=\{B_m : m\in E\}$ *is a subnet of* $\{A_n : n\in D\}\}$, *then*

(1) $\underline{\lim} A_n = \bigwedge_{T\in G} \overline{\lim} B_m$ (2) $\overline{\lim} A_n = \bigvee_{T\in G} \underline{\lim} B_m$

Proof. Using Theorem2. 1 and Theorem2. 8, we have $\underline{\lim} A_n \leqslant \underline{\lim} B_m \leqslant \overline{\lim} B_m$ for every subnet $T=\{B_m : m\in E\}$ of $\{A_n : n\in D\}$, we clearly have $\underline{\lim} A_m \leqslant \bigwedge_{T\in G} \overline{\lim} B_m$.

Conversely, suppose that $e \not\leqslant \underline{\lim} A_m (e\in M)$, then there is a $P\in\eta(e)$ such that for all m in D, we can choose $n(m)\geqslant m$ with $A_{n(m)}\leqslant P$. Let $\{B_m : m\in D\}$ be difined by $B_m = A_{n(m)}$, Then $\{B_m : m\in D\}$ is a subnet of $\{A_n : n\in D\}$ and $B_m\leqslant P$ for each $m\in D$. i. e. $e \not\leqslant \overline{\lim} B_m$. This means that $\bigwedge_{T\in G} \overline{\lim} B_m \leqslant \underline{\lim} A_n$. So the proof of (1) is complete.

Another equality can be similarly proved.

Theorem2. 10. *If every subnet of* $\{A_n : n\in D\}$ *in* $(L(M), \eta)$ *has a subnet converges to* A, *then* $\{A_n : n\in D\}$ *converges to* A.

Proof. Let $T=\{B_m : m\in E\}$ be arbitrary subnet of $\{A_n : n\in D\}$, T has a subnet $\{C_i : i\in F\}$ with $\lim C_i = A$. In the light of theorem2. 8, $\underline{\lim} B_m \leqslant \underline{\lim} C_i = \overline{\lim} C_i = A$, and so $\overline{\lim} A_n = \bigvee_{T\in G} \underline{\lim} B_m \leqslant A$ by Theorem2. 9 and arbitrariness of T. On the other hand, in accordance with Theorem2. 8 we know that $A=\lim C_i = \overline{\lim} C_i \leqslant \overline{\lim} B_m$ and hence $A\leqslant \bigwedge_{T\in G} \overline{\lim} B_m = \underline{\lim} A_n$ on account of Theorem2. 9. Therefore $\underline{\lim} A_n = A$.

3. Some Applications

In this section, we prove some interesting characterizations with regard to closed elements and continuous generalized order—homomorphisms by making use of the convergence theory of L-nets.

Theorem3. 1. *Let* $(L(M), \eta)$ *be a topological molecular lattice and* $A\in L$. *Then the following con-*

ditions are equivalent:

(1) *A is closed*;

(2) $\overline{\lim} A_n \leqslant A$ *for every L-net* $\{A_n : n \in D\}$ *in A*;

(3) $\underline{\lim} A_n \leqslant A$ *for every L-net* $\{A_n : n \in D\}$ *in A*.

Proof. (1)⇒(2): let A be closed element and $\{A_n : n \in D\}$ be an L-net in A. If $e \leqslant \overline{\lim} A_n (e \in M)$, then for each $P \in \eta(e)$ and each $n \in D$ there is an $m \in D$ satisfying $m \geqslant n$ and $A_m \not\leqslant P$. As a result, $A \not\leqslant P$ by virture of the fact that $\{A_n : n \in D\}$ is in A. For this $e \leqslant A^- = A$, and so $\overline{\lim} A_n \leqslant A$.

(2)⇒(3): From Theorem2. 1 and Condition(1) we have $\underline{\lim} A_n \leqslant \overline{\lim} A_n \leqslant A$.

(3)⇒(1): Presume that Condition(3) are true, then for each moleculae $e \in M$ with $e \leqslant A^-$, there exists a molecular net $\{S(n) : n \in D\}$ in which converges to e in line with Theorem4. 22 in[4]. Hence according to Theorem4. 21 in[4] and Condition(3), $e \leqslant A$, i. e. $A^- \leqslant A$ and hence $A = A^-$.

Definition3. 1[4]. Let $f: (L(M), \eta) \longrightarrow (L_1(M_1), \eta_1)$ be a generalized order-homomorphism.

(1) f is said to be continuous if $\forall\ Q \in \eta_1$ we have $f^{-1}(Q) \in \eta$.

(2) f is said to be continuous at point $e \in M$, if $\forall\ Q \in \eta_1(f(e))$ we have $(f^{-1}(Q))^- \in \eta(e)$.

Theorem3. 2. *Suppose that* $f: (L(M), \eta) \longrightarrow (L_1(M_1), \eta_1)$ *is a generalized order-homomorphism. Then the followings are equivalent*:

(1) *f is continuous*;

(2) $\forall\ A \in L, f(A^-) \leqslant (f(A))^-$;

(3) $\forall\ B \in L_1, (f^{-1}(B))^- \leqslant f^{-1}(B^-)$;

(4) $\forall\ e \in M$, *f is continuous at e*;

(5) *For any L-net* $\{A_n : n \in D\}$ *in* $(L(M), \eta)$, $f(\underline{\lim} A_n) \leqslant \underline{\lim} f(A_n)$;

(6) *For every L-net* $\{B_n : n \in D\}$ *in* $(L_1(M_1), \eta_1)$, $\underline{\lim} f^{-1}(B_n) \leqslant f^{-1}(\underline{\lim} B_n)$;

(7) *For every L-net* $\{A_n : n \in D\}$ *in* $(L(M), \eta)$, $f(\overline{\lim} A_n) \leqslant \overline{\lim} f(A_n)$;

(8) *For each L-net* $\{B_n : n \in D\}$ *in* $(L_1(M_1), \eta_1)$ $\overline{\lim} f^{-1}(B_n) \leqslant f^{-1}(\overline{\lim} B_n)$。

Proof. (1)⇔(2)⇔(3): The proof is straightforward and is omitted.

(3)⇒(4): Let $e \in M$ and $P \in \eta(f(e))$. Then by (3) we have $(f^{-1}(P))^- \leqslant f^{-1}(P^-) = f^{-1}(P)$, and hence $f^{-1}(P) = (f^{-1}(P))^- \in \eta(e)$.

(4)⇒(5): Let $\{A_n : n \in D\}$ be an L-net in $(L(M), \eta)$, $e \in M$ and $e \leqslant \underline{\lim} A_n$. Then for each $P \in \eta(f(e))$, $(f^{-1}(P))^- \in \eta(e)$ in line with Condition(4) and Definition3. 1, and then there exists an $n_o \in D$ such that $A_n \not\leq (f^{-1}(P))^-$, specially, $A_n \not\leq f^{-1}(P)$ for all $n \geq n_o$. Since $A_n \not\leq f^{-1}(P)$ implies $f(A_n) \not\leq P$ for all $n \geq n_o$. $f(e) \leq \underline{\lim} f(A_n)$ by Theorem3. 1. This means that $f(\underline{\lim} A_n) \leq \underline{\lim} f(A_n)$

(5)⇒(6): Provided that $\{B_n : n \in D\}$ is an L-net in $(L_1(M_1), \eta_1)$, then from Condition(5) we know that $f(\underline{\lim} f^{-1}(B_n)) \leq \underline{\lim} ff^{-1}(B_n) \leq \underline{\lim} B_n$. Hence $\underline{\lim} f^{-1}(B_n) \leq f^{-1}(\underline{\lim} B_n)$.

(6)⇒(1): Suppose that Q is a closed element in η_1 and that $\{B_n : n \in D\}$ is an L-net in Q. On account of Theorem3. 1 $\underline{\lim} B_n \leq Q$. Hence by using Condition(6) we have $\underline{\lim} f^{-1}(B_n) \leq f^{-1}(\underline{\lim} B_n) \leq f^{-1}(Q)$, and so $f^{-1}(Q)$ is closed in η, in the light of Theorem3. 1 and by Definition3. 1, we know that f is continuous.

(4)⇒(7), (7)⇒(8) and (8)⇒(1) are similar to (4)⇒(5), (5)⇒(6) and (6)⇒(1) respectively. We omit these proofs.

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