Fixed point theorem of fuzzy mapping

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Abstract: in this paper, we prove the fixed point theorem of the mapping from fuzzy metric space to fuzzy metric space, and give its application in the solution of fuzzy differetial equation and fuzzy integral equation

Keywords: Fuzzy metric space, Mapping from fuzzy metric space to fuzzy metric space ; fixed point theorem of fuzzy mapping, fuzzy differential equation, fuzzy integral equation

1: Introduction

Several different approaches to the fuzzy fixed point problem were given in [1, 2], and in [3], Kaleva introduced the differential of the fuzzy functions and gave the solution of the fuzzy differential equation.

In the section 2 of this paper, we give some basic definitions and properties about the fuzzy metric space.

In section 3, we introduce a new mapping which maps from fuzzy metric space to fuzzy metric space, and prove its fuzzy fixed point theorem.

At last, in the section 4, applying the fuzzy fixed point theorem, we prove the solution existence theorem on fuzzy integral equation and fuzzy differetial equation.

2: Fuzzy metric space

Let (X, d) be metric space, P (X) denote the class of all sets in X, F (X) denote the class of all fuzzy sets in X, Fc (X) denote the class of all compact subsets in F(X)

In F(X), we define the metric

$$D(A, B) = \sup d(A_a, B_a) \qquad (1.1)$$

where d is the hausdorff metric in P (X), we call (F (X), D) a fuzzy metric space.

Definition 2.1: Let (F(X), D) be fuzzy metric space, for $X_n, Y \in F(X)$ $(n \geq 1,)$ if $D(X_n, Y) \to 0$ when $n \to \infty$, then we call $\{X_n\}$ converge to X, denoted by $X_n \to Y$, Y is the limit of $\{X_n\}$.

Th2.1 For a convergent sequence $\{X_n\}$ of Fc(X), the limit of $\{X_n\}$ is unique.

Proof: Let Y_1, Y_2 be the limit of $\{X_n\}$, then $0 \leq D(Y_1, Y_2) \leq D(Y_1, X_n) + D(X_n,$

Y_2) and D (X_n, X_1) $\to 0$, D (X_n, Y_2) $\to 0$ when $n \to \infty$, then D (X_1, Y_2) $= 0$ that is $X_1 = Y_2$ complete the proof.

definition2. 2: Let (F(X), D) be a fuzzy metric space, {Xn} be a sequence ofF (X), if for arbitrary $\varepsilon > 0$, there exists a positive number $N(\varepsilon)$, such that,

$$D(X_n, X_m) < \varepsilon$$

holds for n, $m > N(\varepsilon)$, then we call {Xn} a fundamental fuzzy sequence or a cauchy fuzzy sequence.

the following Lemma is trivial

Lemma2. 1: (1): A convergent fuzzy sequence of Fc(X) is a fundamental fuzzy sequence of Fc(X), (2) For a fundamental fuzzy sequence {Xn}in Fc(X), if there exists a subsequence $\{X_{n_k}\}$ of {Xn} such that $\{X_{n_k}\}$ is convergent to Y, then $X_n \to Y$.

Definition2. 3: Let ($\mathcal{L}$, D) be a fuzzy metric space, $\mathcal{L}$ is called complete fuzzy metric space iff every fundamental fuzzy sequence of $\mathcal{L}$ is convergent.

Th2. 2: Let R be real space, F(R) be the family of the fuzzy numbers. then (F(R), D) is a complete fuzzy metric space.

Poof: Let {Xn} ($n \in N$) be a fundamental fuzzy number sequence of F(R), for every $\alpha \in [0,1]$, denote the α-cuts of Xn as

$$(X_n)_\alpha = [(X_n)_\alpha, (X_n)_\alpha]$$

Since {Xn} ($n \in N$) is a fundamental fuzzy sequence of F(R), we can easily verify that $\{(X_n)_\alpha\}$ ($n \in N$) is a fundamental sequence of Pc(R). (where Pc(R) is the family of closed intervals)

That is, for arbitrary $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ such that $D((X_n)_\alpha, (X_m)_\alpha) < \varepsilon$ holds for $n, m > N(\varepsilon)$, therefore,

$$d[(X_n)_\alpha, (X_m)_\alpha] < \varepsilon$$

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hold for n, $m > N(\varepsilon)$, thus $\{(X_n)_\alpha\}$ ($n \in N$) $\{(X_n)_\alpha\}$ ($n \in N$) are two fundamentalsequenceofR, by the classical cauchy convergence principle ofR, $(X_n)_\alpha$ and $(X_n)_\alpha$ converge and we assume that they converge to Xa and Xa respectly, obviously Xa≤Xa, thus {(Xn) a} converge to $X_\alpha = [X_\alpha^-, X_\alpha^+]$ and it is trivial that for every $\beta \in [0,1]$, $\beta > \alpha$, X_β X_α

X_α We define fuzzy set $X(x) = \sup\{\alpha \in [0,1]\ X \in X_\alpha\}$ and obviously $X1 = \{x : X(x) = 1\} \neq \phi$, hence $X \in F(R)$.

Now to prove $X_n \to X$, because for any $\alpha \in [0,1]$, $D[(X_n)_\alpha, X_\alpha] \to 0$ when $n \to \infty$ then

$D(Xn, X) = \sup\{ D((Xn)_{\alpha}, X_{\alpha}\} = 0$

that is $Xn \to X$, hence $F(R)$ is complete .

Th2. 3 Let $C[a, b] = \{f: [a, b] \to Fc(R), f$ is continuous on $[a, b]\}$, then $C[a, b]$ is complete.

Proof: On $C[a, b]$, we define metric:

$$\Delta(f, g) = \sup D(f(x), g(x)) \ .$$

It is obvious that Δ is a metric on $C[a, b]$.

To prove $C[a, b]$ is complete , Let $\{fn\}$ is a fundamental fuzzy sequence in $C[a, b]$, that is, for any $\varepsilon > 0$, there exists $N(\varepsilon)$ such that wherever $n, m > N(\varepsilon)$. $\Delta(fn, fm) = \sup D(fn(x), fm(x)) < \varepsilon$ holds, therefore, for evevry $x \in [a, b]$, we have that $D(fn(x), fm(x)) < \varepsilon$ holds for $n, m > N(\varepsilon)$, this implies that $\{fn(x)\}$ is a fundamental fuzzy number sequence, byTh2. 2, there exists $f(x) \in Fc(R)$ such that the fuzzy sequence $\{fn(x)\}$converge to $f(x)$.

In the following, we are to prove that $f(x) \in C[a, b]$, because $fn(x)$is continuous on $[a, b]$ and $\{fn(x)\}$ isafundamentalfuzzy sequence and converge to $f(x)$, then for arbitrary fixed $x \in [a, b]$

$$\Delta(f(x), f(x\ )) \leq D(f(x), fn(x)) + D(fn(x), fn(x\ )) + D(fn(x), f(x\ )) \to 0$$

when $x \to x$ and $n \to \infty$, that is $f(x)$ is continuous at x , since x is arbitrary on $[a, b]$, then $f(x)$ is continuous on $[a, b]$.

3: Mapping from fuzzy Metric spacetofuzzymetric space and its fuzzy fixed point theorem

Definition3. 1: Let $(\pounds 1, D1)$, $(\pounds 2, D2)$ be two fuzzy metric space, if according to a rule Φ, such that , for every $x \in X$, there exists unique $y \in Y$ correspondence to x, then y is called a fuzzy mapping with respect to x, denoted by $y = \Phi(x)$.

As a special example, if $f: X \to Y$ is a mapping, by Zadeh's extension principle, we can extend f to $F(X) \to F(Y)$ by the equation

$$f(u)(y) = \mathrm{Sup}\ \{ u(x)\}$$

the $f(u)$ is a fuzzy mapping from $F(X)$ to $F(Y)$.

Definition3. 2 A fuzzy mapping Φ: $\pounds 1 \to \pounds 2$ is called continuous at x iff for any $\varepsilon > 0$, there exists $\delta > 0$, such that $D(\Phi(x), \Phi(x\ )) < \varepsilon$ holds for x that satisfies $D(x, x\ ) < \delta$. Φ is called continuous on $\pounds 1$ if for every $x \in \pounds$, Φ is continuous at x.

Definition 3. 3 : Let $(\pounds, D)$ be fuzzy metrric space $\Phi: \pounds \to \pounds$) be a fuzzy mapping. if there exists $k \in [0, 1]$ such that for all $x. y \in \pounds$

$$D(\Phi(x), \Phi(y)) \leq kD(x, y)$$

holds, then Φ is called a contrative fuzzy mapping on $\mathcal{L}$ Th3.2 The contractive fuzzy mapping $\phi: \mathcal{L} \to \mathcal{L}$ is continuous.

Proof: Trivial

Definition3.4: Let $T: \mathcal{L} \to \mathcal{L}$ be a fuzzy mapping, a point $x \in \mathcal{L}$ satisfying $Tx = x$ is called a fuzzy fixed point of T.

Th3.2: Let $(\mathcal{L}, D)$ be a complete fuzzy metric space and a fuzzy mapping $T: \mathcal{L} \to \mathcal{L}$ be a contractive fuzzy mapping, then the mapping T has unique fuzzy fixed point.

Proof: Let x_0 be a arbitrary element of $\mathcal{L}$ denote:

$$x_1 = Tx_0, \ x_2 = Tx_1 = T^2 x_0, \ \ldots \ x_n = Tx_{n-1} = T^n x_0 \ldots$$

We can verify that the sequence $\{x_n\}$ $(n \in N)$ is fundamental. In fact, by the definition3.3

$$D(x_{n+1}, x_n) = D(Tx_n, Tx_{n-1}) < \alpha D(x_n, x_{n-1})$$

where $0 \leq \alpha < 1$, $n \geq 1$.

by the mathematical induction, we can easily obtain:

$$D(x_{n+1}, x_n) \leq \alpha^n D(x_1, x_0), \quad (n \geq 1) \tag{3.1}$$

then for arbitrary positive integer p, by triangle inequality of D and equation (2.1), we have:

$$D(x_{n+p}, x_n) \leq D(x_{n+p}, x_{n+p-1}) + D(x_{n+p-1}, x_{n+p-2}) + \cdots + D(x_{n+1}, x_n)$$

$$\leq (\alpha^{n+p-1} + \alpha^{n+p-2} + \ldots + \alpha^n) D(x_1, x_0)$$

$$= \frac{\alpha^n - \alpha^{n+p}}{1-\alpha} D(x_1, x_0) \leq \frac{\alpha^n}{1-\alpha} D(x_1, x_0) \to 0$$ when $n \to \infty$, thus $\{x_n\}$ is a fundamental fuzzy sequence on $\mathcal{L}$, because

$\mathcal{L}$ is complete, so there exists unique $x^* \in \mathcal{L}$, such that $X_n \to x^*$.

Since the T is continuous, then $Tx_n \to Tx^*$ and $Tx_n = x_{n+1} \to x^*$, then

$D(Tx^*, x^*) \leq D(Tx^*, Tx_n) + D(Tx_n, x^*) \to 0$ (as $n \to \infty$), hence $D(Tx^*, x^*) = 0$ then $Tx^* = x^*$

therefore x^* is a fuzzy fixed point of T.

To prove the uniqueness, suppose x' is another fuzzy fixed point of T. that is $Tx' = x'$. since $D(x, x') \leq D(Tx, Tx') \leq \alpha D(x, x')$ and $0 \leq \alpha < 1$, the above inequality holds iff $D(x, x') = 0$ holds that is $x = x'$.

Complete the proof.

4: The applications of fuzzy fixed point theorem

(1): Integral of fuzzy function.

Definition4. 1: Let F: [a. b]→F(R), the integral of F(t) on [a, b] is defined by the following equation:

$$[\int_a^b F(t)dt] = \int_a^b F(t)dt = \{\int_a^b f(t)dt \mid f: [a,b]\to R \text{ is a measurable slector of } F(t)\}.$$

here $0< \alpha \leq 1$ F(t) is called integrable on [a, b] iff $\int_a^b F(t)dt \in F(R)$.

Osmo Kaleva [1] has proved that the continuous fuzzy funtions are integrable.

(2): On fuzzy integral equation of Fredholm.

In the following, we deal with the fuzzy integral equation which has the following form:

$$\Phi(x) = \lambda\int_a^b K(x,y)\Phi(y)dy + f(x) \qquad (4.1)$$

Where a, b are finite real numbers, λ is real number, f (x) is a continuuous fuzzy function on [a, b] , and K [x, y] is a continuous fuzzy function on [a, b] $\times$ [a. b].

Define K: C[a, b]→C[a, b] by $(K\Phi)(x) = \int_a^b K(x,y)\Phi(y)dy$

Th4. 1 If K (x, y) satisfies the conditions: for arbitrary $x\in[a,b]$, there exists real number $M>0$, such that

$$\Delta(K\Phi, K\Phi) \leq M\Delta(\Phi, \Phi)$$

then . for $|\lambda| \leq 1/M$. the equation (3. 1) has unique solution in F(R).

Proof: Let $(T\Phi)(x) = f(x) + \lambda\int_a^b K(x,y)\Phi(y)dy$.

$$\Delta(K\Phi, K\Phi) = \Delta(f+\lambda K\Phi, f+\lambda k\Phi) = \sup\{D(f(x) + \lambda\int_a^b K(x,y)\Phi(y)dy, f(x) + \lambda\int_a^b K(x,y)\Phi(y)dy)\}$$

$$= |\lambda| \sup D(\int_a^b K(X,y)\Phi(y)dy, \int_a^b K(x,y)\Phi(y)dy) \leq |\lambda| M\Delta(\Phi, \Phi).$$

when $|\lambda| < 1/M$, themapping K is a contractive fuzzy mapping. by Th3. 2 , the mapping K has unique fuzzyfixed point x , and Kx = x , that is x is the unique solution of

(4.1).

Following from Th3.2 and Th4.1, we can obtain the approximate solution of (4.1) as folows.

For a given fuzzy function f_0, Let

$$f_1(x)=f0(x)+(\lambda Kf_0(x)=f_0(x)+\lambda\int_a^b K(x,y)f_0(y)dy\in F(R)$$

$$f_2(x)=f_1(x)+(\lambda Kf_1(x)=f_1(x)+\lambda\int_a^b K(x,y)f_1(y)dy\in F(R)$$

....

$$f_n(x)=f_{n-1}(x)+(\lambda Kf_{n-1}(x)=f_{n-1}(x)+\lambda\int_a^b K(x,y)f_{n-1}(y)dy\in F(R)$$

....

Example1: \For a given fuzzy integral equation:

$$\Phi(x)=a+\lambda\int_0^1 xy\Phi(y)dy \qquad (4.2)$$

where a is a positive fuzzy number, to find the solution, we let $f_0(x)=1$

$$f1(x)=a+\lambda\int_0^1 xydy=a+\frac{1}{2}\lambda xy$$

$$f2(x)=a+\lambda\int_0^1 xy\ f1(x)dx=a+\frac{\lambda xy}{2}+\frac{\lambda^2 x}{6}$$

$$f3(x)=a+\lambda\int_0^1 xyf\ (x)dx=a+\frac{a\lambda x}{2}+\frac{\lambda^2 x}{6}+\frac{\lambda^3 x}{18}$$

......

$$fn(x)=a+\int_0^1 xyf\ (x)dx=a+\frac{a\lambda x}{2}\sum_{n=0}^{n-1}\frac{\lambda^n}{3^n}+\frac{\lambda^n}{2\times 3^n}x$$

as $\lambda<3$, the fuzzy function series $\{f_n(x)\}$ converge to $\Phi(x)=a+\frac{a\lambda x}{2}\sum_0^\infty\frac{\lambda^n}{3^n}=a+3\frac{\lambda ax}{2(3-\lambda)}$, $\Phi(x)$ is the solution of the equation (4.2)

3: On fuzzy differrential equation.

For a fuzzy differential equation:

$$\frac{d\Phi(x)}{dx}=f(x,\Phi(x)) \qquad (4.3)$$

$$\Phi(x_0)=y_0$$

Where f is a classcal continuous function with two variables, and f (x , y) is defined by Zadeh's Extension Principle, $y_0\in F(R)$ is a fuzzy contant number.

By the theorem in 1, the equation (4.3) is equal to the equation

$$\Phi(x) = y_0 + \int_{x0}^{x} f(t, \phi(t))dt \tag{4.4}$$

Theorem4.2 If the mapping f satisfies the following conditions:

For every $t \in [a, b]$, there exists a positive number L such that:

$$\Delta(\int_{x0}^{x} f(t, \Phi(t)dt, \int_{x0}^{x} f(t, \Phi(t)dt) \leq L\ \Delta(\Phi(t), \Phi(t))$$

then when $\lambda < 1/L$, the equation (4.3) has unique solution $\Phi^{*}(x) \in F(R)$

Proof: Define fuzzy mapping $T: C[a, b] \to C[a, b]$ by the following equation:

$$(T\Phi)(X) = y_0 + \int_{x0}^{x} f(t, \Phi(t))dt$$

Since $\Delta(T\Phi\ , T\Phi\ ) = \Delta(y_0 + \lambda\int_{x0}^{x} f(t, \Phi(t))dt,\ y_0 + \lambda\int_{x0}^{x} f(t, \Phi(t))dt) = \Delta(\lambda \int_{x0}^{x} f(t, \Phi(t))dt,\ \lambda\int_{x0}^{x} f(t, \Phi(t))dt) = \lambda\,\Delta(\int_{x0}^{x} f(t, \Phi(t)), \int_{x0}^{x} f(t, \Phi(t))dt) \leq \lambda\ \Delta$
$(\Phi\ \ \Phi)$

Since $\lambda < 1/L$, that is $\lambda\ L < 1$. Therefore the fuzzy mapping T iscontractive fuzzy mapping, by Th3.2, it has unique fuzzy fixed point $\Phi^{*}(x)$, $\Phi^{*}(x)$ is the solution of equation (4.3), and the solution isunique.

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