FUNCTIONAL SYSTEM IN FUZZY LOGIC FORMAL THEORY

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Abstract

The formal theory of fuzzy logic suggested by V. Novak [1] is considered. The class of all logical functions is described.

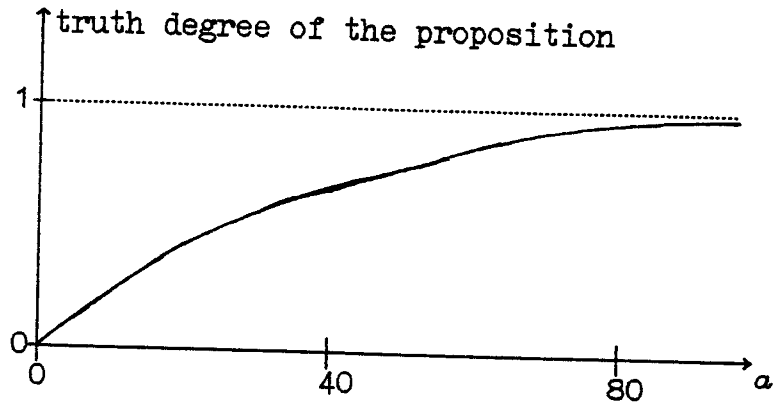
Introduction.

This work is devoted to the description of the functional system in fuzzy logic. The formal theory of first-order fuzzy logic is suggested by V. Novak [1]. Here we consider fuzzy propositional calculus which is closely connected with the functional system of fuzzy logic. Namely, the functional system is a semantic component of fuzzy propositional calculus . As concerns first-order fuzzy logic, its semantics is more complex and must include not only functional interpretations of logical connectives but also interpretations of functional and predicate symbols.

Fuzzy logic is a generalization of classical one. The main difference between fuzzy and classical logic is in the set of truth values. There are only two truth values in classical logic: 0 (*false*) and 1 (*true*). Fuzzy logic has more than two truth values. That's why fuzzy logic is also called by many-valued logic. Usually the set of truth values is denoted by L. In this work we consider continuum-valued logic whose truth value set is the interval $[0,1]$.

In practice such an extension of the set of truth values is necessary when it is difficult to determine whether a proposition is true or false. For example we consider the proposition "*Mr X is old*". Let us denote the age of Mr X by a. If a is small (close to 0) then X is a child and there is no doubt that the proposition is false, otherwise if a is large (more than 80) then the proposition is true . But what should we decide if $a=30$ or $a=50$? We could agree to consider that X is old if a is more than some threshold number but such an agreement discords with a continuous character of getting older.

Fuzzy logic suggests more natural way of description for the properties like "*to be old*". Namely, we use the truth values between 0 and 1, i.e between *false* and *true*. In our example we can consider that X is old in degree $1-exp(-a/40)$ where a is the age of Mr X:



Of course, other variants of truth valuation for this proposition are possible.

Fuzzy logic as well as classical one operates with constructions of a special language named formulas. The language of a propositional calculus consists of proposition variables, logical connectives and auxiliary symbols (parentheses).

A proposition variable is one whose values may be propositions. We assume that proposition variables form a countable set and use the notations $X_1, X_2, \ldots$ for them. A proposition variable is a simple (atomic) formula.

Logical connectives are the symbols which join simple formulas into complex ones. There are the following logical connectives in classical logic:

- binary: $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication);
- unary: $\neg$ (negation);

In comparison with the classical case fuzzy propositional calculus is enriched by the symbols of proposition constants $\{a, a \in L\}$ and the binary connective $\&$.

Formulas define as usual:

(a) All the proposition variables and constants are formulas;
(b) If A and B are formulas then $\neg A$ and $(A * B)$ are formulas where $*$ is a binary connective.

For example, $\neg(X_1 \rightarrow (X_2 \vee X_1))$ is a formula but $\&(X_1 \neg X_2)$ is not a formula.

Usually the unnecessary parentheses are omitted. For example, $X_1 \wedge X_2$ is usually written instead $(X_1 \wedge X_2)$.

Let A be a formula. Let us write $A(X_1, \ldots, X_n)$ if only $X_1, \ldots, X_n$ have appearances in A and the other proposition variables have no appearance in A.

The definition of truth functions is a generalization of analogous definition in classical logic. Namely, we introduce the algebraic operations on L corresponding to each of the logical connectives. Considering L with these operations we obtain the algebra of truth values

$\mathbf{L}=\langle L,\wedge,\vee,\neg,\otimes,\rightarrow,\{a,a\in L\}\rangle$

where

- $\wedge$ (conjunction), $\vee$ (disjunction), $\otimes$ (multiplication), $\rightarrow$ (implication) are binary operations;
- $\neg$ (negation) is an unary operation;
- $\{a,a\in L\}$ are constants.

We consider the following functional interpretation of logical connectives suggested by V. Novak in [1]:

$a\wedge b=min(a,b)$;
$a\vee b=max(a,b)$;
$a\otimes b=(a+b-1)^*$;
$a\rightarrow b = (1-a+b)^*$;
$\neg a=1-a$,

$$\text{where } x^*=\begin{cases} 0 \text{ if } x<0, \\ x \text{ if } x\in[0,1] \\ 1 \text{ if } x>1. \end{cases}$$

Let $A=A(X_1,\dots,X_n)$ be a formula. Then the truth function of A is a function $f_A:L^n\rightarrow L$ such that $f_A(\mathbf{x})=f_A(x_1,\dots,x_n)$ is the value of the expression obtained from A by replacing X_i with x_i $(1\le i\le n)$ and each logical connective with the corresponding operation.

For example, $f_A(x_1,x_2)=\neg(x_1\rightarrow(x_2\vee x_1))$ is the truth function for the formula $A(X_1,X_2)=\neg(X_1\rightarrow(X_2\vee X_1))$. Note that $f_A(x_1,x_2)=0$ for any $x_1,x_2\in L$. It means that the formula is always false.

Let us denote the class of truth functions of all formulas by H and call it by the class of fuzzy logic functions. It follows from the definition that this class consists of all superpositions of the operations $\wedge,\vee,\neg,\otimes,\rightarrow$ and the constants.

In classical logic as well as in finite-valued logic the class of all truth functions coincides with the class of all operations on the set of truth values. Two questions arise:

1. Does the situation remain in case of $L=[0,1]$?
2. In case it does not, what properties must have an operation on L to be a truth function for some formula?

Evidently, the first question have the negative answer. Indeed, the cardinality of H is continuum while the class of all operations on L is more than continuum. Hence, there are operations on L which are not fuzzy logic functions.

The second question is more difficult. The presented work suggests the answer of this question.

The work consists of four parts. The Part 1 contains the main definitions which are necessary for understanding of the obtained results and their proofs. The Part 2 contains the formulation of the main result obtained in this work. Parts 3,4 contain the proof of the main result.

1. Main Definitions.

Let L be a set of truth values. Assume here that $L=[0,1]$. Define the following operations (connectives) on L (see also [1]):

- binary: $\wedge$ (conjunction), $\vee$ (disjunction), $\otimes$ (multiplication), $\rightarrow$ (implication);
- unary: $\neg$ (negation);
- nullary: (constants) $\{a, a\in L\}$,

where

$a\wedge b=min(a,b)$;
$a\vee b=max(a,b)$;
$a\otimes b=(a+b-1)^*$;
$a\rightarrow b = (1-a+b)^*$;
$\neg a=1-a$;

$$x^*=\begin{cases} 0 \text{ if } x<0, \\ x \text{ if } x\in[0,1], \\ 1 \text{ if } x>1. \end{cases}$$

So we define the algebra $\mathbf{L}$: $\mathbf{L}=\langle L,\wedge,\vee,\neg,\otimes,\rightarrow,\{a,a\in L\}\rangle$.

Definition 1. A set $\mathbf{D}\subseteq\mathbb{R}^n$ is a (convex) polyhedron if there are (m,n)-matrix $\mathbf{A}$ and m-vector $\mathbf{b}$ so that $\mathbf{D}=\{\mathbf{x}\in\mathbb{R}^n \mid \mathbf{Ax}\le\mathbf{b}\}$.

Let us assume that the elements of $\mathbf{A}$ are integers and the elements of $\mathbf{b}$ are real.

Definition 2. A function $f:L^n\rightarrow L$ is piecewise linear if there are $r\in\mathbb{N}$ and polyhedra $\mathbf{D}_1,\ldots,\mathbf{D}_r$ such that $\bigcup_{i=1}^{r}\mathbf{D}_i=L^n$ and $f|_{\mathbf{D}_1},\ldots,f|_{\mathbf{D}_r}$ are linear functions, i.e. $f(\mathbf{x})=\mathbf{c}^T\mathbf{x}+d$, $\mathbf{x}\in\mathbf{D}_i$, $1\le i\le r$,

for some column-vector $\mathbf{c}$ with integer elements and real number d.

We will name the polyhedra $\mathbf{D}_1,\ldots,\mathbf{D}_r$ by linearity domains of function f.

Note that all the piecewise linear functions are continuous.

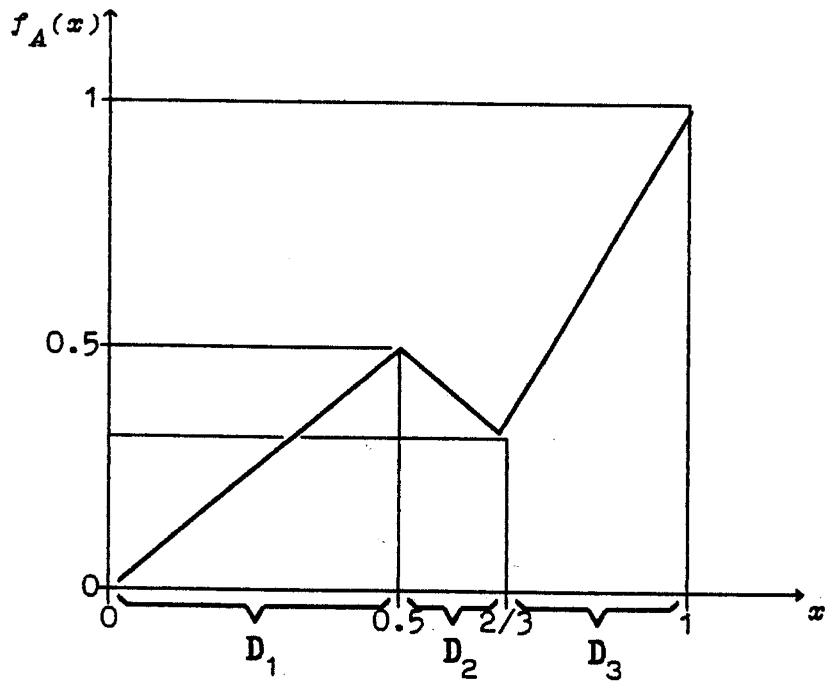
Let us denote the class of all n-ary piecewise linear functions by K_n.

By the definition K_0 contains of constants. We define $K=\bigcup_{n=0}^{\infty}K_n$.

Definition 3. Let M be a set of operations on L. M is closed class if it is closed with respect to the superposition.

The closed class, generated by the operations from $\mathbf{L}$, will be denoted by H.

Example of a piecewise linear function of one variable:



D_1, D_2 and D_3 are the linearity domains for f_A;

$D_1=\{X\in\mathbb{R} \mid A_1X\le b_1\}$. $D_2=\{X\in\mathbb{R} \mid A_2X\le b_2\}$. $D_3=\{X\in\mathbb{R} \mid A_3X\le b_3\}$;

$A_1=A_2=A_3=\begin{bmatrix}-6\\6\end{bmatrix}$, $b_1=\begin{bmatrix}0\\3\end{bmatrix}$, $b_2=\begin{bmatrix}-3\\4\end{bmatrix}$, $b_3=\begin{bmatrix}-4\\6\end{bmatrix}$;

$f_A(X)=C_1X+d_1$, $X\in D_1$, $f_A(X)=C_2X+d_2$, $x\in D_2$, $f_A(X)=C_3X+d_3$, $x\in D_3$;

$C_1=1$, $d_1=0$, $C_2=-1$, $d_2=1$, $C_3=2$, $d_3=-1$.

Definition 3. Let M be a set of operations on L. M is closed class if it is closed with respect to the superposition.

The closed class, generated by the operations from L, will be denoted by H.

2. The Main Result.

The main result of this work is:

Theorem 1. The class of all piecewise linear functions is equal to the closed class of operations from L, i.e. $K=H$.

3. The Class of Piecewise Linear Functions.

Proposition 1. All the operations from L are piecewise linear functions.□

Lemma 1. Let $f_1,\dots,f_p\in K_n$. Then there are the polyhedra $D_1,\dots,D_r$ *such that* $\bigcup_{i=1}^{r} D_i=L^n$ and $D_1,\dots,D_r$ are the linearity domains for each of $f_1,\dots,f_p$.

Proof. Let $D_{j1},\dots,D_{jr_j}$ are the linearity domains of f_j, $1\le j\le p$. For any p-tuple $s=(i_1,\dots,i_p)$ *where* $1\le i_1\le r_1,\dots,1\le i_p\le r_p$ let us denote

$$D_s=D_{i_1\dots i_p}=D_{1i_1}\cap\dots\cap D_{pi_p}.$$

D_s is the polyhedron as it is the intersection of polyhedra. It is obvious that D_s is the linearity domain for each of $f_1,\dots,f_p$ and $\bigcup_{s} D_s=L^n$.□

Proposition 2. K is closed with respect to the superposition.

Proof. (a). It is obvious that all the identity operations are piecewise linear functions.

(b). Let $f_1,\dots,f_p\in K_n$, $g\in K_p$, $n,p\ge 1$. We'll prove that $h\in K_n$ where $h(\mathbf{X})=g(f_1(\mathbf{X}),\dots,f_p(\mathbf{X}))$.

Based on lemma 1 suppose that $f_1,\dots,f_p$ have common linearity domains $D_1,\dots,D_r$, such that $\bigcup_{i=1}^{r} D_i=L^n$. Let $D'_1,\dots,D'_q$ be linearity domains of g and $\bigcup_{k=1}^{q} D_k=L^p$. Consider the mapping $f:L^n\to L^p$:

$$f(\mathbf{X})=(f_1(\mathbf{X}),\dots,f_p(\mathbf{X}))^T.$$

It is easy to check that $D_{ik}=D_i\cap f^{-1}(D'_k)$ is the polyhedron and $g\circ f|_{D_{ik}}$ is the linear function. Hence, $h=g\circ f$ is the piecewise linear function.□

The following proposition is the corollary of the propositions 1 and 2:

Proposition 3. H is the subset of K.□

4. The Class of Fuzzy Logic Functions.

Using the denotation L^{L^n} for the set of all mappings from L^n to L, let us denote $H_n=L^{L^n}\cap H$.

For $D\subseteq L^n$ define the class of functions $H_D\subseteq L^{L^n}$:

$$H_D=\{f\in L^{L^n} \mid \exists h\in H_n : h|_D=f|_D\}.$$

Obviously, $H_{L^n}=H_n$.

Let $n\geq 1$. Define the class $\mathbf{D}_n$ contains of sets $D\subseteq L^n$ as:

$$D=\{\mathbf{X}\in L^n \mid g(\mathbf{X})=1\},\ g\in H_n.$$

Obviously, $L^n\in\mathbf{D}_n$.

Lemma 2.

(a). Let $D_1,D_2\in\mathbf{D}_n$. Then $D_1\cup D_2\in\mathbf{D}_n$ and $H_{D_1\cup D_2}=H_{D_1}\cap H_{D_2}$.

(b). Let $D_1,\ldots,D_r\in\mathbf{D}_n$, $\bigcup_{i=1}^{r} D_i=L^n$. Then $f\in H_{D_i}$, $1\leq i\leq r$, implies $f\in H_n$.

Proof. (a). Let $D_1=\{\mathbf{X}\in L^n \mid g_1(\mathbf{X})=1\}$, $D_2=\{\mathbf{X}\in L^n \mid g_2(\mathbf{X})=1\}$, $g_1,g_2\in H_n$. Then $D_1\cup D_2=\{\mathbf{X}\in L^n \mid g_1(\mathbf{X})\vee g_2(\mathbf{X})=1\}$, whence $D_1\cup D_2\in\mathbf{D}_n$.

The inclusion $H_{D_1\cup D_2}\subseteq H_{D_1}\cap H_{D_2}$ is obviously true. We'll prove now that the converse inclusion is true also.

Let $f\in H_{D_1}\cap H_{D_2}$. Then there exist $h_1,h_2\in H_n$ such that $f|_{D_1}=h_1|_{D_1}$, $f|_{D_2}=h_2|_{D_2}$. Let k and l are some natural numbers. Define the function $h:L^n\to L$ as:

$$h(\mathbf{X})=(h_1(\mathbf{X})\otimes(g_1(\mathbf{X}))^l)\vee(h_2(\mathbf{X})\otimes(g_2(\mathbf{X}))^k),$$

where $y^m \overset{\text{def}}{=} y\otimes y\otimes\ldots\otimes y$ (m times), $y\in L$.

Obviously, $h\in H_n$. It can be proved that $f|_{D_1\cup D_2}=h|_{D_1\cup D_2}$ when k and l are sufficiently large. It means that $f\in H_{D_1\cup D_2}$. Hence, $H_{D_1\cup D_2}=H_{D_1}\cap H_{D_2}$.

(b) follows from (a) by induction.□

Define the functions $s_n^k: L^n \to L$, $n\geq 1$, $k\in Z$, $k\geq 0$:

$$s_n^k(\mathbf{X})=s_n^k(x_1,\ldots,x_n)=(\sum_{i=1}^{n} x_i-k)^*.$$

Obviously, $\oplus=s_2^1$. We'll use the notation $\circledast$ for s_2^0. As $a\circledast b=(\neg a)\to b$, then $\circledast\in H_2$.

Lemma 3. $s_n^k\in H_n$, $n\geq$, $k\geq 0$.

Proof by induction.

(a). $s_1^0=\delta_1^1\in H_1$.

(b). Assume that $s_{n-1}^k\in H_{n-1}$, $k\geq 0$. Let $k\geq 1$, $\mathbf{X}=(x_1,\ldots,x_n)^T$, $\mathbf{X}'=(x_1,\ldots,x_{n-1})^T$. Then $s_n^0(\mathbf{X})=x_1\circledast\ldots\circledast x_n$;

$$s_n^k(\mathbf{X})=\begin{cases}(x_n+s_{n-1}^{k-1}(x')-1)^*=\delta_n^n(x)\circledast s_{n-1}^{k-1}(x') \text{ if } \neg s_{n-1}^k(\mathbf{x}')=1,\\ (x_n+s_{n-1}^{k}(x')-1)^*=\delta_n^n(x)\circledast s_{n-1}^{k}(x') \text{ if } s_{n-1}^{k-1}(\mathbf{x}')=1.\end{cases}$$

Obviously, $s_n^0\in H_n$. Using the inductive assumption and Lemma 2(a) we obtain $s_n^k\in H_n$.□

Lemma 4. Let $f\in L^{L^n}$, $f(\mathbf{X})=f(x_1,\ldots,x_n)=(\sum_{i=1}^{n} a_i\mathbf{X}_i+b)^*$, $a_i\in Z$, $1\leq i\leq n$. Then $f\in H_n$.

Proof. Let $\mathbf{X}=(x_1,\ldots,x_n)^T\in L^n$ and

$$x_i'=\begin{cases}x_i \text{ if } a_i\geq 0\\ \neg x_i \text{ if } a_i<0\end{cases};$$

$$a_i'=|a_i|;\ b'=\{b+\sum_{i,a_i<0} a_i\};\ k=[b+\sum_{i,a_i<0} a_i],$$

where $1\leq i\leq n$, $[c]$ is the integer part of c, i.e. the maximal integer not greater than c, $\{c\}=c-[c]$, $c\in\mathbb{R}$.

Then

$$f(\mathbf{X})=(\sum_{i=1}^{n} a_i'x_i'+b'-k)^*,$$

where $a_i'\geq 0$, $1\leq i\leq n$, $b'\in[0,1]$ and $k\in Z$.

Let $q=\sum_{i=1}^{n} a_i'+1$. If $k<0$ then $f\equiv 1\in H$. Otherwise f is the superposition of functions δ_n^i, negation $\neg$, constant b′ and function s_q^k.□

Now we can prove the main theorem 1 (see p.2), which proposes that $K=H$. Due to the proposition 3, it is sufficient to show, that $K\subseteq H$.

Proof of theorem 1. Let $f\in K_n$, D is the linearity domain of f and $D=\{X\in R^n | AX\leq b\}$ where A is the integer $(m*n)$-matrix and b is the m-vector. Let $a_1,\ldots,a_m$ are the rows of A. Define the functions $g_1,\ldots,g_m,g\in L^{L^n}$ as follows:

$g_i(X)=(a_iX-b)^*$, $1\leq i\leq m$; $g(X)=\neg(g_1(X)\vee\ldots\vee g_m(X))$.

Due to lemma 4, $g_1,\ldots,g_m,g\in H$. But $D=\{X\in R^n | g(X)=1\}$. Hence $d\in D_n$. It follows from lemma 4 that $f\in H_D$ for any linearity domain D. Using lemma 2 we obtain $f\in h_n$. It proves that $K\subseteq H$. □

References.

1. V. Novak. On the syntactico-semantical completeness of first-order fuzzy logic. Part 1. Kybernetika 26(1990), 1, pp. 47-66.