A Theorem on the Contrapositive Symmetry Property of the Mean Triangle Products of Relations

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First we state some basic facts concerning the presence and absence of the property of contrapositive property for implication operators that are needed in the sequel. This is followed by the section on relational products and the proof of the theorem concerning the interrelationship of the mean products computed by EZ, W and KD implication operators.

1 Contrapositivity of Implication Operators

1.1 Basic Notions

Contrapositivity of the implication operator is one of the basic properties of classical 2-valued (Boolean, crisp) logic. We say that a fuzzy implication operator $\rightarrow_i$ possesses *contrapositive symmetry* iff $(a \rightarrow_i b) = (1-b) \rightarrow_i (1-a)$.

Not all the many-valued operators that default into crisp logic on the values {0, 1}, retain this property for the whole range of their values. The following results are well known [1],[2]:

(1.1) Operators $\rightarrow_2$ (S: Standard Strict), $\rightarrow_5$ (L: Lukasiewicz), $\rightarrow_6$ (KD: Kleene-Dienes) possess contrapositive symmetry.
(1.2) Operators $\rightarrow_1$ (S#: Standard Sharp), $\rightarrow_3$ (S*: Standard Star, Gödel), $\rightarrow_4$ (G43: Gaines-Goguen) **lack** it.

1.2 Relevance for Applications

If the data that may be processed by fuzzy logic is not contrapositive, then using a ply operator that has the contrapositive symmetry property leads to the distortion of the results of fuzzy computations with that data. The results are contaminated by the mathematical artifact, namely by the contrapositive symmetry that was not present in the original data.

This fact is often ignored, but it may lead to serious distortions of analysis of scientific data.

Let us illustrate the significance of this effect by an example from the domain of medicine. If the presence of symptom a implies the presence of symptom b, then the absence of the symptom b implies the absence of the symptom a when the property of contrapositive symmetry holds.

One can describe the absence of disease by the absence of certain signs and symptoms. Then the concepts of "health" and "disease" are also symmetric. This is, however **not** true universally.

If, on the other hand, the property of contrapositive symmetry does not hold, it is not possible any more to infer the absence of the symptom a from the absence of the symptom b.

Because in a number medical problems the contrapositive property of data does **not** hold universally, using a contrapositive operator on non-contrapositive medical data may produce misinformation that may have very serious consequences for the patient. Hence, to test for contrapositivity symmetry property of medical, and indeed, of any scientific data is a **very important** epistemological requirement.

1.3 Upper and Lower Contrapozitivization

Interesting theoretically, and important for applications of fuzzy logics and relations, is the relationship between non-contrapositive implication (ply) operators and the ply operators that can be generated from these by the operations of **contrapositivization**.

Definition [3]: *lower* and *upper contrapositivization.*

Given an implication operator $\rightarrow_i$ lacking the property of contrapositive symmetry, we construct from it implication operators that possess it by the following formulas:

(2.1) **lower contrapositivization** $\rightarrow_{i'}$: $(a \rightarrow_{i'} b) = (a \rightarrow_i b) \wedge (1-b) \rightarrow_i (1-a)$

(2.2) **upper contrapositivization** $\rightarrow_{i''}$: $(a \rightarrow_{i''} b) = (a \rightarrow_i b) \vee (1-b) \rightarrow_i (1-a)$

1.3.1 On Early-Zadeh, Kleene-Dienes and Wilmott Implication Operators

Let us define following:

For $\wedge$ and $\vee$ we take *min* and *max* connectives, respectively; EZ (Early Zadeh ply): $a \rightarrow_{EZ} b = (a \wedge b) \vee (1-a)$; KD (Kleene-Dienes): $a \rightarrow_{KD} b = (1-a) \vee b$; W (Willmott): $a \rightarrow_W b = (a \rightarrow_{EZ} b) \wedge \kappa a$; crispness $\kappa b = b \vee (1-b)$.

The EZ operator, which lacks the contrapositive symmetry itself, can be used to generate new operators having the property. This can be done by *lower* or *upper* contrapositivization of EZ. Thus we have a theorem that will be needed in Sec.2:

Theorem: Bandler and Kohout [3]:

W and KD operators are the lower and the upper contrapositivization of EZ, respectively.

For theorems concerning other implication operators see e.g. [10].

2 Non-Contrapositive Relational Compositions

2.1 A Brief Summary of BK-Products

The triangle *sub-product* $R \lhd S$, the triangle *super-product* $R \rhd S$, and *square product* $R \square S$ were first introduced by Bandler and Kohout in 1977, and are referred to as the BK-products in the literature. Their theory and applications have made substantial progress since then. See the survey in [9] with a list of 50 selected references on the mathematical theory and applications of BK-products in various fields of science and engineering.

Mathematical definitions: Where R is a relation from X to Y, and S a relation from Y to Z, a *product relation* $R * S$ is a relation from X to Z, determined by R and S. There are several types of product used to produce product-relations [5], [9]. Each product type performs a **different logical action** on the intermediate sets, as *each logical type* of the product enforces a *distinct specific meaning* on the resulting product-relation $R * S$. We have the following definitions of the products. In these definitions, R_{ij}, S_{jk} represent the fuzzy degrees to which the respective statements x_iRy_j, y_jSz_k are true.

PRODUCT TYPE	SET-BASED DEFINITION	MANY-VALUED LOGIC FORMULA
Circle product:	$x(R \circ S)z \Leftrightarrow xR$ intersects Sz	$(R \circ S)_{ik} = \bigvee_j (R_{ij} \bigwedge S_{jk})$
Triangle Subproduct:	$x(R \lhd S)z \Leftrightarrow xR \tilde{\subseteq} Sz$	$(R \lhd S)_{ik} = \bigwedge_j (R_{ij} \rightsquigarrow S_{jk})$
Triangle Superproduct:	$x(R \rhd S)z \Leftrightarrow xR \tilde{\supseteq} Sz$	$(R \rhd S)_{ik} = \bigwedge_j (R_{ij} \leftarrow S_{jk})$
Square product:	$x(R \Box S)z \Leftrightarrow xR \cong Sz$	$(R \Box S)_{ik} = \bigwedge_j (R_{ij} \equiv S_{jk})$

Families of fuzzy products of the same logical type that have to be distinguished. The customary logical symbols for the logic connectives *AND, OR*, both *implications* and the *equivalence* in the above logic formulas represent the connectives of some many-valued logic. When the relations are fuzzy, there is a wide choice of realization for each of the four product kinds, because a number of many-valued logic *implication operators*. The details of choice of the appropriate many-valued connectives are discussed in [3],[4],[9].

It is important to distinguish *harsh* fuzzy products (using MIN[a, b] as the outer operator) from a different family, the family of *mean* products. Given the general formula $(R@S)_{ik} ::= \#(R_{ij} * S_{jk})$, a mean product is obtained by replacing the outer connective # by $\sum$ and normalizing the resulting product appropriately. The mean products have provided an effective inference mechanism in CLINAID applications [8] and have been shown also to be important in other applications of relational products [9],[6],[4],[11],[7],[12].

2.2 Theorems on the dependency of the Mean Triangle Products Computed by EZ, KD and W Implication Operators

Those relational compositions that use an implication (ply) operator as their inner operation will inherit the property or its absence from their respective ply operator. Hence, the epistemological problems that arise from the lack of this property carry over also to the non-symmetry with respect to negation of relational compositions. Before we proceed with the proof of the theorem, we briefly summarize some known facts on triangle products of relations.

2.3 A Theorem on Contrapositive Symmetry of Relational Products

We have seen that the EZ (Early Zadeh) implication operator, which lacks the contrapositive symmetry itself, can be used to generate new operators having the property. This yields Willmott implication operator as the *lower* contrapositivization of EZ, and Kleene-Dienes KD implication operator as the *upper* contrapositivization of EZ.

This has a consequence for the interrelationship of the mean triangle products computed by these three implication operators. Thus we have the following theorem.

Theorem 2.1:

If the negative (positive) in their arguments triangle mean products computed by KD and EZ are identical, then the positive products (negative) products computed by W and EZ are identical. In symbols:

(a) If $(\neg S \lhd \neg R)_{KD} = (\neg S \lhd \neg R)_{EZ}$ holds, then $(R^{-1} \lhd S^{-1})_W = (R^{-1} \lhd S^{-1})_{EZ}$ holds also.

(b) If $(S \triangleleft R)_{KD} = (S \triangleleft R)_{EZ}$ holds, then $(\neg R^{-1} \triangleleft \neg S^{-1})_W = (\neg R^{-1} \triangleleft \neg S^{-1})_{EZ}$ holds also.

Proof:
Assume that KD and EZ give same negative product. i.e.,

$$(\neg S \triangleleft \neg R)_{KD} = (\neg S \triangleleft \neg R)_{EZ}$$

Then, for all elements of both products it is valid that

$$(\neg S \triangleleft^{KD} \neg R)_{ik} = (\neg S \triangleleft^{EZ} \neg R)_{ik}, \quad \forall i \in I, \forall k \in K$$

By the definition of the product we get

$$\frac{1}{N_j}\sum_j(\neg S_{ij} \rightarrow^{KD} \neg R_{jk}) = \frac{1}{N_j}\sum_j(\neg S_{ij} \rightarrow^{EZ} \neg R_{jk}), \quad \forall i \in I, \forall j \in J, \forall k \in K$$

Then,

$$\frac{1}{N_j}\sum_j(\neg S_{ij} \rightarrow^{KD} \neg R_{jk}) = \frac{1}{N_j}\sum_j min((\neg S_{ij} \rightarrow^{KD} \neg R_{jk}), max(\neg S_{ij}, S_{ij}))$$

by the definition of EZ operator.
Since the operator of KD is contrapositive symmetric,
$(\neg S_{ij} \rightarrow^{KD} \neg R_{jk}) = (R_{jk} \rightarrow^{KD} S_{ij}) = (R^{-1}_{kj} \rightarrow^{KD} S^{-1}_{ji})$
and $(R^{-1}_{kj} \rightarrow^{KD} S^{-1}_{jk}) \leq max(S_{ij}, \neg S_{ij})$. Also,

$$\frac{1}{N_j}\sum_j(R^{-1}_{kj} \rightarrow^{W} S^{-1}_{ji}) = \frac{1}{N_j}\sum_j min((R^{-1}_{kj} \rightarrow^{KD} S^{-1}_{ji}), max(R^{-1}_{kj}, \neg R^{-1}_{kj}), max(S^{-1}_{ji}, \neg S^{-1}_{ji}))$$

by the definition of W operator.
Since $min((R^{-1}_{kj} \rightarrow^{KD} S^{-1}_{ji}), max(S^{-1}_{ji}, \neg S^{-1}_{ji})) = (R^{-1}_{kj} \rightarrow^{KD} S^{-1}_{ji})$,

$$\frac{1}{N_j}\sum_j(R^{-1}_{kj} \rightarrow^{W} S^{-1}_{ji})$$

=

$$\frac{1}{N_j}\sum_j min((R^{-1}_{kj} \rightarrow^{KD} S^{-1}_{ji}), max(R^{-1}_{kj}, \neg R^{-1}_{kj}))$$

=

$$\frac{1}{N_j}\sum_j(R^{-1}_{kj} \rightarrow^{EZ} S^{-1}_{ji})$$

by the definition of EZ operator.
So, $(R^{-1} \triangleleft^{W} S^{-1})_{ik} = (R^{-1} \triangleleft^{EZ} S^{-1})_{ik} \quad \forall i \in I, \forall k \in K$.
Therefore, $(R^{-1} \triangleleft S^{-1})_W = (R^{-1} \triangleleft S^{-1})_{EZ}$. This completes the proof.
The proof for the case of the positive products can be argued in a similar way.

We can also ask whether a similar equality holds for KD ply operator, given that the relational products computed by EZ (a non-contrapositive ply) and W (a contrapositive ply) are equal.

The answer is that the equality is not valid, but it is satisfiable. That means that there exist relations for which it holds, but one can find other relations for which this does not hold. It is easy to prove the following theorem.

Theorem 2.2:

The statement "The negative (positive) triangle mean products computed by EZ and KD are identical to degree α, if the positive (negative) products computed by EZ and W ply operators are the same to degree α, respectively" is **satisfiable**, *but* **not valid** *universally.*

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