

Correlation and information energy of interval valued fuzzy numbers

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Abstract In this paper, we introduce the correlation coefficient of interval valued fuzzy numbers, and study their some properties. Furthermore, we present the concept of information energy, and compare interval valued fuzzy numbers with them.

Keywords Interval valued fuzzy numbers; correlation coefficient; information energy.

Introduction

Interval valued fuzzy sets were suggested at first by Grozafczany B(1) and Turksen B(2). They had been applied to the fields of Approximate inference, Signal transmission and controller etc. Nevertheless, in the course of applying to practice, the membership function of a fuzzy set isn't easy to be determined usually. But the membership degree of an interval valued is relative easy to be determined. So how to compare interval valued fuzzy sets is an important task in theory of interval valued fuzzy sets and its application. In this paper, we first give the concept of interval valued fuzzy numbers, correlation and information energy. And then we shall conclude and compare interval valued fuzzy numbers with them.

1 Interval valued and interval valued fuzzy sets

Throughout this paper, let I be a closed unit interval, i. e., $I = [0, 1]$

Let $[I] = \{\bar{a} = [a^-, a^+]: a^- \leq a^+, a^-, a^+ \in I\}$, Especially for arbitrary $a \in I$, put $a = [a, a]$ Then $a \in [I]$ is obvious. For any $\bar{a}, \bar{b} \in [I]$, let $\bar{a} = [a^-, a^+]$, $\bar{b} = [b^-, b^+]$ define $\bar{a} \wedge \bar{b} = [a^- \wedge b^-, a^+ \wedge b^+]$, $\bar{a} \vee \bar{b} = [a^- \vee b^-, a^+ \vee b^+]$, $\bar{a}' = [1 - a^+, 1 - a^-]$ Furthermore, we define the ordered in $[I]: \bar{a} \leq \bar{b}$ iff $a^- \leq b^-, a^+ \leq b^+$.

In particularly, we have

$\bar{a} = \bar{b}$ iff $a^- = b^-, a^+ = b^+$; $\bar{a} < \bar{b}$ iff $\bar{a} \leq \bar{b}$ and $\bar{a} \neq \bar{b}$

clearly, $([I], \leq, \vee, \wedge)$ constitutes a complete lattice with a minimal element $\bar{0} = [0, 0]$ and a maximal element $\bar{1} = [1, 1]$.

And then, we introduce partial orders \subset .

Let $\bar{a}, \bar{b} \in [I]$, define $\bar{a} \subset \bar{b}$ iff $b^- < a^- \leq a^+ < b^+$;

Definition 1.1 Let X be an ordinary set, mapping $A: X \rightarrow [I]$ is called an interval valued fuzzy set on X .

Let $IF(X)$ denote all interval valued fuzzy sets on X . For each $A \in IF(X)$. Let $A(x) = [A^-(x), A^+(x)]$. $A^-(x) \leq A^+(x)$, where $x \in X$. Then ordinary fuzzy sets $A^-: X \rightarrow I$ and $A^+: X \rightarrow I$ are called a lower fuzzy set of A and a upper fuzzy set of A respectively.

Simply write $A = [A^-, A^+]$.

Especially, A is called a degenerate ordinary set, whenever $A^-(x) = A^+(x) \equiv 1$ or $A^-(x) = A^+(x) \equiv 0$

For any $A, B \in IF(X)$, $x \in X$. We define

$(A \cap B)(x) = A(x) \wedge B(x)$ $(A \cup B)(x) = A(x) \vee B(x)$.

$A'(x) = (A(x))' = [1 - A^+(x), 1 - A^-(x)] = [A^{+'}(x), A^{-'}(x)]$

Especially, $A = B$ iff $A^-(x) = B^-(x)$, $A^+(x) = B^+(x)$. for all $x \in X$

Definition 1.2 Let $A \in IF(X)$, arbitrary $[\lambda_1, \lambda_2] \in [I]$, ordinary sets

$$A_{[\lambda_1, \lambda_2]} = \{x \in X: A^-(x) \geq \lambda_1, A^+(x) \geq \lambda_2\};$$

$$A_{(\lambda_1, \lambda_2)} = \{x \in X: A^-(x) > \lambda_1, A^+(x) > \lambda_2\};$$

$A_{[\lambda_1, \lambda_2]} = \{x \in X: A^-(x) \geq \lambda_1, A^+(x) > \lambda_2\}$ and $A_{(\lambda_1, \lambda_2)} = \{x \in X: A^-(x) > \lambda_1, A^+(x) \geq \lambda_2\}$ are called $[\lambda_1, \lambda_2]$ - level set of A, (λ_1, λ_2) - strong level set of A, $[\lambda_1, \lambda_2)$ - right Strong level Set of A and $(\lambda_1, \lambda_2]$ - left strong level Set of A respectively. Where (λ_1, λ_2) , $[\lambda_1, \lambda_2)$ and $(\lambda_1, \lambda_2]$ are not intervals. They are only signals. Here we allow $\lambda_1 = \lambda_2$. Clearly, we have $A_{[0,0]} = X$,

$$A_{[1,1]} = A_{(1,1)} = A_{(1,1)} = \emptyset, \quad A_{(0,0)} = A_{(0,0]} \subset A_{[0,0]} \subset A_{[0,0]}, \quad A_{[\lambda_1, \lambda_2]} = A_{\lambda_1}^- \cap A_{\lambda_2}^+,$$

$$A_{(\lambda_1, \lambda_2)} = A_{\lambda_1}^- \cap A_{\lambda_2}^+, \quad A_{[\lambda_1, \lambda_2)} = A_{\lambda_1}^- \cap A_{\lambda_2}^+, \quad A_{(\lambda_1, \lambda_2]} = A_{\lambda_1}^- \cap A_{\lambda_2}^+.$$

$$\text{where } A_{\lambda_1}^- = \{x \in X: A^-(x) > \lambda_1\}, \quad A_{\lambda_2}^+ = \{x \in X: A^+(x) > \lambda_2\}$$

2. Interval valued fuzzy numbers and their correlation

Definition 2.1 Let R be the set of all real numbers. $A \in IF(R)$. If the following conditions are satisfied:

- (1) A is normal. i. e., there is at least an $x_0 \in R$ such that $A(x_0) = [1, 1]$;
- (2) A is convex, i. e., $A(\lambda x + (1 - \lambda)y) \geq \min(A(x), A(y))$, for each $x, y \in R$ and $\lambda \in [0, 1]$;
- (3) A^- and A^+ are upper semi - continuous;
- (4) The cloure of $A_{(0,0)}$ is compact.

Then A is called an interval valved fuzzy number on R. And we denote all interval valued fuzzy numbers on R as $IF^*(R)$

proposition 2.1 $A \in IF^*(R)$ iff A^- and A^+ are ordinary fuzzy numbers.

proposition 2.2 If $A \in IF^*(R)$. Then for any $[\lambda_1, \lambda_2] \in [I] - \{[0, 0]\}$, $A_{[\lambda_1, \lambda_2]}$ is a closed interval on R.

The proof refers to [6]

Lemma 2.1 If A be an ordinary fuzzy number. Then there exists a closed interval $[\delta_1, \delta_2]$, such that $A_0 \subset [\delta_1, \delta_2]$.

The proof refers to (6)

Theorem 2.2 If $A \in IF^*(R)$. Then there exists a closed interval $[a, b]$, such that $A_{(0,0)} \subset [a, b]$
proof. By proposition 2.1, we can get that A^- and A^+ are ordinary fuzzy numbers.

By lemma 2.1, we know that there exists closed intervals $[\delta_1, \delta_2]$ and $[\delta_1', \delta_2']$, such that $A_0^- \subset [\delta_1, \delta_2]$, $A_0^+ \subset [\delta_1', \delta_2']$

Hence by the normal of A, there exists an $x_0 \in R$, such that $A^-(x_0) = A^+(x_0) = 1 > 0$

i. e., $x_0 \in A_0^- \cap A_0^+$. Then $A_0^- \cap A_0^+ \neq \emptyset$

Let $a = \delta_1 \wedge \delta_1', b = \delta_2 \vee \delta_2'$. We obtain $A_{(0,0)} = A_0^- \cap A_0^+ \subset [a, b]$.

Because of it, for given an interval valued fuzzy number $A \in IF^*(R)$, we need only consider the interval valued fuzzy numbers defined on some closed interval $[a, b]$.

Let $IF^*([a, b]) = \{A: A \text{ is an interval valued fuzzy number with } A_{(0,0)} \subset [a, b]\}$

Definition 2.2 let $A, B \in IF^*([a, b])$. We define

$$m(A, B) = [m^-(A, B), m^+(A, B)] \quad \text{where}$$

$$m^-(A, B) = \frac{1}{(b-a)} \int_a^b (A^-(x)B^-(x) + A^{-'}(x)B^{-'}(x)) dx,$$

$$m^+(A, B) = \frac{1}{(b-a)} \int_a^b (A^+(x)B^+(x) + A^{+'}(x)B^{+'}(x)) dx.$$

$$\text{Furthermore, we call } \rho(A, B) = \left[\frac{m^-(A, B)}{\sqrt{m^-(A, A)m^-(B, B)}}, \frac{m^+(A, B)}{\sqrt{m^+(A, A)m^+(B, B)}} \right]$$

an interval corelation coefficient with respect to A and B.

It is easy to show $0 \leq A^-(x)B^-(x) + A^{-'}(x)B^{-'}(x)$

$$= 1 - A^-(x)(1 - B^-(x)) - B^-(x)(1 - A^-(x)) \leq 1$$

Similarly, $0 \leq A^+(x)B^+(x) + A^{+'}(x)B^{+'}(x) \leq 1$ for all $x \in [a, b]$

property 1. If $A, B \in IF^*([a, b])$. Then we have

$$(1) m(A, A) \geq [0, 0]; \quad (2) [0, 0] \leq m(A, B) \leq [1, 1];$$

$$(3) m(A, B) = m(B, A), \quad \rho(A, B) = \rho(B, A)$$

property 2. If $A \in IF^*([a, b])$. Then $m(A, A) = [1, 1]$ iff A is a degenerate ordinary set.

proof. Sufficiency is obvious.

Necessity: Suppose A isn't a degenerate ordinary set. Then there is at least an $x_0 \in [a, b]$,

such that $A(x_0) = [\lambda_1, \lambda_2]$ and $\lambda_1 < \lambda_2 \leq 1$. i. e., $A^-(x_0) = \lambda_1 < \lambda_2 = A^+(x_0)$.

We choose η_1 and η_2 which fulfill $\lambda_1 < \eta_1 < \lambda_2$ and $\lambda_2 < \eta_2 < 1$.

Clearly, we have $[\lambda_2, \lambda_2] < [\eta_2, \eta_2] \in [I]$.

By the normal of A, we can get

$$A_{[\eta_1, \eta_2]} = A_{\eta_1}^- \cap A_{\eta_2}^+ \neq \emptyset \text{ and } A_{\eta_1}^-, A_{\eta_2}^+ \text{ are both closed intervals on } \mathbb{R}.$$

$$\text{we let } A_{\eta_1}^- = [c', d'], A_{\eta_2}^+ = [c'', d''], c = c' \vee c'', d = d' \wedge d''. S = [a, c] \cup [d, b]$$

Then they follow that $A_{[\eta_1, \eta_2]} = [c, d]$ and $x_0 \notin [c, d]$

Thus $x_0 \in S$. i. e., $S \neq \emptyset$.

Therefore, for any $x \in S$, $A(x) < [\eta_1, \eta_2]$ holds.

Without loss of generality, we assume that $0 < A^-(x) < \eta_1 < 1$

$$\text{Then } (A^-(x))^2 + (A^{-'}(x))^2 = 2(A^-(x) - \frac{1}{2})^2 + \frac{1}{2} < 1.$$

Hence, we obtain

$$\begin{aligned} m^-(A, A) &= \frac{1}{b-a} \int_a^b ((A^-(x))^2 + (A^{-'}(x))^2) dx \\ &= \frac{1}{b-a} \left(\int_a^c ((A^-(x))^2 + (A^{-'}(x))^2) dx + \int_c^d ((A^-(x))^2 + (A^{-'}(x))^2) dx \right) \\ &< \frac{1}{b-a} \left(\int_a^c dx + \int_c^d dx \right) = 1. \text{ This is contradiction with assumption.} \end{aligned}$$

property 3. Let $A, B \in IF^*([a, b])$. Then $[0, 0] \leq \rho(A, B) \leq [1, 1]$.

proof. First, from $m(A, B) \geq 0$ implies directly $\rho(A, B) \geq [0, 0]$.

Second, from Schwarz inequality,

$$\text{we have } m^-(A, A) \cdot m^-(B, B) = \frac{1}{(b-a)^2} \int_a^b ((A^-(x))^2 + (A^{-'}(x))^2) dx \cdot \int_a^b ((B^-(x))^2 + (B^{-'}(x))^2) dx$$

$$\begin{aligned}
&\geq \frac{1}{(b-a)^2} \left| \int_a^b ((A^-(x))^2 + (A^{-'}(x))^2)^{\frac{1}{2}} \cdot ((B^-(x))^2 + (B^{-'}(x))^2)^{\frac{1}{2}} dx \right|^2 \\
&= \frac{1}{(b-a)^2} \left| \int_a^b ((A^-(x))^2 (B^-(x))^2 + (A^{-'}(x))^2 (B^{-'}(x))^2 + (A^-(x))^2 (B^{-'}(x))^2 + \right. \\
&(A^{-'}(x))^2 (B^-(x))^2)^{\frac{1}{2}} dx \left. \right|^2 \\
&\geq \left| \frac{1}{b-a} \int_a^b ((A^-(x)B^-(x) + A^{-'}(x)B^{-'}(x))^{\frac{1}{2}} dx \right|^2 = (m^-(A, B))^2 \\
&\text{Similarly, } m^+(A, A) \cdot m^+(B, B) \geq (m^+(A, B))^2. \text{ Thus we have } [0, 0] \leq \rho(A, B) \leq [1, 1]
\end{aligned}$$

property 4. Let $A, B \in IF^*([a, b])$. Then $\rho(A, B) = [1, 1]$ iff $A = B$

proof. Sufficiency is obvious. By property 3, we can implies necessity. So omit it

property 5. Let $A, B \in IF^*([a, b])$. Then $m(A, B) = [0, 0]$ iff A and B are degenerate ordinary sets.

proof. Sufficiency is obvious.

Necessity: If $\frac{1}{b-a} \int_a^b (A^-(x)B^-(x) + A^{-'}(x)B^{-'}(x)) dx = 0$

Since integrand function is non-negative and upper semi-continuous.

Then it follows that $A^-(x)B^-(x) + A^{-'}(x)B^{-'}(x) \equiv 0$ holds, for all $x \in [a, b]$.

Which implies that $A^-(x)B^-(x) = 0$ and $A^{-'}(x)B^{-'}(x) = 0$. Hence, following facts hold:

If $A^-(x) = 0$, then $A^{-'}(x) = 1$ so that $B^{-'}(x) = 0$ and $B^-(x) = 1$

Similarly, if $B^-(x) = 0$, we shall get $A^-(x) = 1$

Therefore by definition 1.1, A and B are degenerate ordinary sets.

Corollary 5. $\rho(A, B) = [0, 0]$ iff A and B are degenerate ordinary sets.

3. The information energy of interval valued fuzzy numbers.

Definition 3.1 Let $A \in IF(X)$, any $x \in X$. Then A is called an interval valued fuzzy set which its fuzzy degree is maximum, if $A(x) \equiv [\frac{1}{2}, \frac{1}{2}]$; A is called an interval valued fuzzy set which its fuzzy degree is minimum, if $A(x) \equiv [0, 0]$. At this time, A is a degenerate ordinary set.

For $A \in IF^*([a, b])$, any $x \in [a, b]$

$$\text{since, } \frac{1}{2} \leq (A^-(x))^2 + (A^{-'}(x))^2 = 2(A^-(x) - \frac{1}{2})^2 + \frac{1}{2} \leq 1,$$

$$\frac{1}{2} \leq (A^+(x))^2 + (A^{+'}(x))^2 = 2(A^+(x) - \frac{1}{2})^2 + \frac{1}{2} \leq 1$$

$$\text{or } 0 \leq 2(A^-(x))^2 + 2(A^{-'}(x))^2 - 1 \leq 1, \quad 0 \leq 2(A^+(x))^2 + 2(A^{+'}(x))^2 - 1 \leq 1$$

So we let $E(A^-, x) = 2(A^-(x))^2 + 2(A^{-'}(x))^2 - 1,$

$$E(A^+, x) = 2(A^+(x))^2 + 2(A^{+'}(x))^2 - 1,$$

$$E(A, x) = [E(A^-, x) \wedge E(A^+, x), E(A^-, x) \vee E(A^+, x)]$$

Note: where $E(A^-, x) \leq (A^+, x)$ doesn't certain hold.

Then we call $E(A, x)$ the interval valued information energy of A at x .

obviously, we have $[0, 0] \leq E(A, x) \leq [1, 1]$

Definition 3.2 Let $A \in IF^*([a, b])$, $E_{[a, b]}(A^-) = \frac{1}{b-a} \int_a^b E(A^-, x) dx,$

$$E_{[a,b]}(A^+) = \frac{1}{b-a} \int_a^b E(A^+, x) dx.$$

$$E_{[a,b]}(A) = [E_{[a,b]}(A^-) \wedge E_{[a,b]}(A^+), E_{[a,b]}(A^-) \vee E_{[a,b]}(A^+)]$$

Then we call $E_{[a,b]}(A)$ the interval valued information energy of A on $[a, b]$ obviously, we have $[0, 0] \leq E_{[a,b]}(A) \leq [1, 1]$ and $E_{[a,b]}(A) \in [I]$

Theorem 3.1 Let $A \in IF^*([a, b])$. Then $E_{[a,b]}(A) = [0, 0]$ iff A is an interval valued fuzzy set which its fuzzy degree is maximum.

proof. Sufficiency is obvious.

Necessity: By $E_{[a,b]}(A) = [0, 0]$, we can get $E_{[a,b]}(A^-) = E_{[a,b]}(A^+) = 0$

$$\text{If } \frac{1}{b-a} \int_a^b (2(A^-(x))^2 + 2(A'^-(x))^2 - 1) dx = 0. \text{ i.e., } \int_a^b (2A^-(x) - 1)^2 dx = 0$$

Then it follows that $(2A^-(x) - 1)^2 \equiv 0$. i.e., $A^-(x) \equiv \frac{1}{2}$, any $x \in [a, b]$

Similarly, for any $x \in [a, b]$, we can prove $A^+(x) \equiv \frac{1}{2}$.

Therefore $A(x) \equiv [\frac{1}{2}, \frac{1}{2}]$. Consequently necessity holds.

Theorem 3.2. Let $A \in IF^*([a, b])$. Then $E_{[a,b]}(A) = [1, 1]$ iff A is a degenerate ordinary set.

proof. Sufficiency: If $A^-(x) = A^+(x) = 1$. Then $A'^-(x) = A'^+(x) = 0$;

If $A^-(x) = A^+(x) = 0$ Then $A'^-(x) = A'^+(x) = 1$.

Hence, however we have $(A^-(x))^2 + (A'^-(x))^2 = (A^+(x))^2 + (A'^+(x))^2 = 1$.

$$\text{i.e., } E_{[a,b]}(A^-) = E_{[a,b]}(A^+) = \frac{1}{b-a} \int_a^b dx = 1$$

$$\text{Necessity: If } \frac{1}{b-a} \int_a^b (2(A^-(x))^2 + 2(A'^-(x))^2 - 1) dx = 1$$

$$\text{Then } \int_a^b ((A^-(x))^2 + (A'^-(x))^2 - 1) dx = 0.$$

For arbitrary $x \in [a, b]$, by $(A^-(x))^2 + (A'^-(x))^2 - 1 = 2A^-(x)(A^-(x) - 1) \leq 0$ and A^- is upper semi-continuous,

we obtain $(A^-(x))^2 + (A'^-(x))^2 - 1 \equiv 0 \quad x \in [a, b]$

i.e., $2A^-(x)(A^-(x) - 1) \equiv 0$ Consequently, $A^-(x) \equiv 0$ or $A^-(x) \equiv 1$

Similarly, $A^+(x) \equiv 0$ or $A^+(x) \equiv 1$. i.e., A is a degenerate ordinary set.

Definition 3.3. Let $A, B \in IF^*([a, b])$.

If $E_{[a,b]}(A) \leq E_{[a,b]}(B)$. Then we call B stronger than A , denote $A \leq B$;

Especially, if $E_{[a,b]}(A) = E_{[a,b]}(B)$. Then we call B equals A , denote $A = B$;

If $E_{[a,b]}(A) < E_{[a,b]}(B)$, Then we call B absolute stronger than A , denote $A < B$.

Definition 3.4. Let $A, B \in IF^*([a, b])$. If $E_{[a,b]}(A) \subset E_{[a,b]}(B)$. Then we call B wholly includes A . Written $A \subset B$.

Example 1. Let $A, B \in IF^*([0, 6])$. Look at picture 1 given by

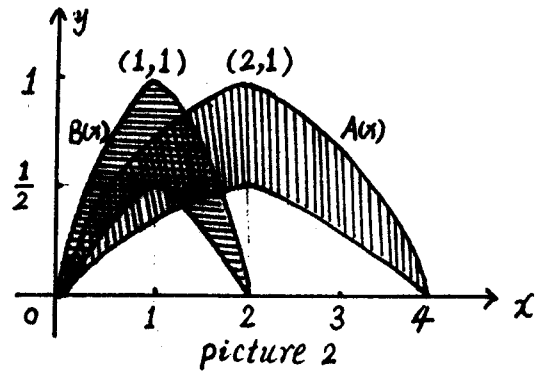
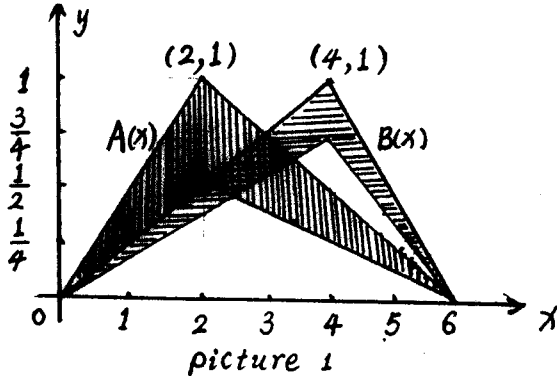
$$A(x) = \begin{cases} [\frac{1}{4}x, \frac{1}{2}x] & 0 \leq x \leq 2 \\ [\frac{3}{4} - \frac{x}{8}, \frac{3}{2} - \frac{1}{4}x] & 2 < x \leq 6 \end{cases} \quad \text{and}$$

$$B(x) = \begin{cases} [\frac{3}{16}x, \frac{1}{4}x] & 0 \leq x \leq 4 \\ [\frac{9}{4} - \frac{3}{8}x, 3 - \frac{1}{2}x] & 4 < x \leq 6 \end{cases}$$

By calculating, we obtain $E_{[0,6]}(A^-) = \frac{1}{3}$, $E_{[0,6]}(A^+) = \frac{1}{3}$, $E_{[0,6]}(A) = [\frac{1}{3}, \frac{1}{3}]$,

$$E_{[0,6]}(B^-) = \frac{1}{4}, \quad E_{[0,6]}(B^+) = \frac{1}{3}, \quad E_{[0,6]}(B) = [\frac{1}{4}, \frac{1}{3}].$$

Thus $E_{[0,6]}(B) \leq E_{[0,6]}(A)$. therefore A is stronger than B, i.e., $B \leq A$.



Example 2. Let $A, B \in IF^*([0, 4])$. Look at picture 2 given by

$$A(x) = [-\frac{1}{8}(x-2)^2 + \frac{1}{2}, -\frac{1}{4}(x-2)^2 + 1] \quad 0 \leq x \leq 4 \text{ and}$$

$$B(x) = \begin{cases} [-\frac{1}{2}(x-1)^2 + \frac{1}{2}, -(x-1)^2 + 1] & 0 \leq x \leq 2 \\ [0, 0] & 2 < x \leq 4 \end{cases}$$

By calculating, we obtain $E_{[0,4]}(A^-) = \frac{1}{5}$, $E_{[0,4]}(A^+) = \frac{7}{15}$, $E_{[0,4]}(A) = [\frac{1}{5}, \frac{7}{15}]$

$$E_{[0,4]}(B^-) = \frac{3}{5}, \quad E_{[0,4]}(B^+) = \frac{11}{15}, \quad E_{[0,4]}(B) = [\frac{3}{5}, \frac{11}{15}].$$

Therefore $E_{[0,4]}(A) < E_{[0,4]}(B)$. Then B is absolute stronger than A. i.e., $A < B$.

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