FIXED POINT THEOREM FOR FUZZY MAPPINGS IN H-SPACE

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Abstract: This paper studied the fuzzy property on H-Space, Bring forward the concept of fuzzy mappings and concept of fixed point of fuzzy mappings on H-Space, and studied fixed point theorems, The results presented unify and extend schauder's fixed point thoorem、Browder's fixed point theorem and some recent important results.

Keywords: H-Space, Fuzzy mapping, Fixed point theorem, Schauder's fixed point theorem, Browder's fixed point theorem.

0 INTRODUCTION

In 1988, C. Bardaro and R. Ceppitelli [1] bring forward the concept of H-Space and studied fixed point theorems. In 1994, Chang Shisheng and Xiang Shuwen [3] bring forward the concept of locally H-convex spaces and studied fixed point theorems, This paper studied the fuzzy property on H-Space, Bring forward the concept of fuzzy mappings and concept of fixed point of fuzzy mappings on H-Space, The results presented in this paper unify and extend schauder's fixed point theorem、Browder's fixed point theorem and some recent important results.

1 PRELIMINARIES

DEFINITION 1. 1 An H-Space is a pair $(X, \{P_A\})$, where X is a topological space and $\{\Gamma_A\}$ is a given family of nonempty contractible subsets of X, indexed by the finite subsets of X, such that $A \subset B$ implies $\Gamma_A \subset \Gamma_B$.

DEFINITION 1. 2 Let $(X, \{\Gamma_A\})$ be an H-Space, A subset $D \subset X$ is called H-Convex relative to subset $C \subset X$ if, for every finite subset $A \subset C$, it follows $\Gamma_A \subset D$. When $C = D$, then D is called H-convex briefly.

DEFINITION 1. 3 An H-Space $(X, \{\Gamma_A\})$ is called locally H-convex space, if for every $\varepsilon > 0$ and every open neighboruhood U of x, there exists a open neighboruhood V of x such that U is H-convex relative to V.

DEFINIFION 1. 4 Let (X, d) be a metric space, A nonempty subset $D \subset X$ is called uniformly locally H-Convex subset, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x \in D$, $B_\varepsilon(x)$ is H-Convex relative to $B_\delta(x)$, where $B_r(x) = \{y \in X, d(x, y) < r\}$.

REMARK 1. 5 It is easy to prove that locally convex topological vector spaces is locally H-Convex spaces, normed spaces is uniformly locally H-Convex spaces.

DEFINITION 1. 6 Let X is a topological space, D is a nonempty subset of X, A mapping $B: D \to [0, 1]$ is called a fuzzy subset over D, we denote by F (D) the family of all fuzzy subsets over D.

DEFINITION 1. 7 A mapping T from $D \to F(D)$ is called fuzzy mapping Over D, for each $x \in D$, $T(x) = T_x \in F(D)$, i. e., T_x is a fuzzy set over D.

DEFINITION 1. 8 Let $T: D \to F(D)$, T is said to be a fuzzy function iff there exists only one $z_x \in D$ such that $T_x(z_x) = \max_{\mu \in D} T_x(\mu) > 0$.

DEFINITION 1. 9 A fuzzy function $T: D \to F(D)$ is said to be a very fuzzy function, iff $T_x(z_x) > 0$ and $\forall u \neq z_x$, $T_x(\mu) = 0$.

DEFINITION 1. 10 If $T: D \to F(D)$ is a fuzzy function, then by using T we can define $T: {}'D \to F(D)$ as follows:

$$T'_x(\mu) = \begin{cases} T_x(z_x) & \mu = z_x \\ 0 & \mu \neq z_x \end{cases}, \quad \forall \mu \in D$$

It is obvious that $T: {}'D \to F(D)$ is a very fuzzy function.

DEFINITION 1. 11 Let $T: D \to F(D)$, $A \in F(D)$, Then the image of A by T is the fuzzy set $T[A] \in F(D)$ defined by:

$$T[A](y) = \sup_{x \in D} \{T_x(y) \cdot A(x)\} \quad \forall y \in D$$

The inverse—image of A by T is the fuzzy set $T^{-1}[A] \in F(D)$ defined by:

$$T^{-1}[A](x) = \sup_{y \in D} \{T_x(y) \cdot A(y)\} \quad \forall x \in D$$

If A is the usual subset of D, then A (x) is the characteristic function of A.

DEFINITION 1. 12 The fuzzy mapping T is F-continuous iff for any $x \in D$ and for any open set C in X we have: $T_x \subseteq C \cap D$ implies that $\exists\ V_x \in V(x)$ with $T[V_x] \subseteq C \cap D$ where V (x) denotes the set of all neighbourhoods of x in X.

DEFINITION 1. 13 Let $T: D \to F(D)$, $x_0 \in D$, if $T_{x_0}(x_0) = \max_{\mu \in D} T_{x_0}(\mu)$, Then x_0 is called a fixed point of T.

LEMMA 1. 14 ([2], [5]) Let $(X, \{\Gamma_A\})$ is an H-Space, $x_1, x_2, \cdots, x_n$ is points of X (not necessarily distinct): Then for a standard $(n-1)$ —Simplex $e_1\ e_2 \cdots e_n$ there exists a continuous mapping $f: e_1\ e_2 \cdots e_n \to X$ such that

$$f(e_{i_1}, \cdots, e_{i_k}) \subset \Gamma(x_{i_1}, \cdots, x_{i_k})$$

Where $\{i_1, i_2, \cdots i_k\}$ is any nonempty subset of $\{1, 2, \cdots, n\}$.

LEMMA 1. 15 ([3]) Let (X, d) is a matric space, $(X, \{\Gamma_A\})$ is a locally H-Convex

space, D is a compact subset of X, Then D is a uniformly locally H-Convex subset of X
proof. $\forall\ \varepsilon>0$ for each $x\in D$, since $(X, \{\Gamma_A\})$ is a locally H-Convex space, therefore there exists $\delta_x\in(0,\varepsilon)$ such that $B_{\frac{\varepsilon}{2}}(x)$ is H-Convex relative to $B_{\delta_x}(x)$, i. e. $\forall$ a finite subset $A\subset B_{\delta_x}(x)$, it follows $\Gamma_A\subset B_{\frac{\varepsilon}{2}}(x)$.

Since D is a compact subset of X, and $D\subset\bigcup_{x\in D} B_{\frac{\delta_{x_i}}{2}}(x)$, hence there exists a finite subset $\{x_1, \cdots, x_n\}\subset D$ such that $D\subset\bigcup_{i=1}^{n} B_{\frac{\delta_{x_i}}{2}}(x_i)$.

Choose $\delta=\min\{\frac{\delta_{x_i}}{2}, i=1, 2, \cdots, n\}$, for each $x\in D$ $\forall$ a finite subset $A\subset B_\delta(x)$, $\because$ $x\in D$, $\therefore\ \exists\ i\in\{1, 2, \cdots, n\}$ wish that $x\in B_{\delta\frac{x_i}{2}}(x_i))$ $\because A\subset B_\delta(x)\subseteq B_{\frac{\delta_{x_i}}{2}}(x)\subset B_{\frac{\delta_{x_i}}{2}}(B_{\frac{\delta x_i}{2}}(x_i))\subseteq B_{\delta_{x_i}}(x_i)$, $\therefore\ \Gamma_A\subset B_{\frac{\varepsilon}{2}}(x_i)\subset B_{\frac{\varepsilon}{2}}(B_{\frac{\delta_{x_i}}{2}}(x))\subset B_\varepsilon(x)$, this completes the proof.

2 MAIN RESULTS

THEOREM 2. 1 Let (X, d) is a matric space, $(X, \{\Gamma_A\})$ is a locally H-Convex space, D is nonempty compact subset of X and D is H-Convex, Let $T: D\to F(D)$ is a fuzzy function, if $T': D\to F(D)$ is F-Continuous, Then there exists $x_0\in D$, x_0 is a fixed point of T.

Proof. since $T: D\to F(D)$ is a fuzzy function, therefore $\forall\ x\in D$ there exists only one $z_x\in D$ with $T_x(z_x)=\max_{\mu\in D} T_x(\mu)>0$, by difinition 1. 10, $T'_x(z_x)>0$, $T'_x(\mu)=0\ \forall\ \mu\neq z_x$. $\forall\ x\in D$ let $f(x)=z_x$, we shall show that $f: D\to D$ is continuous, $\forall\ x_0\in D$, if open set $C\subset X$ such that $f(x_0)=z_{x_0}\in C$, then $C(z_0)=1$, by difinition 1. 9、1. 10, $T'_{x_0}(z_{x_0})>0$, $T'_{x_0}(u)=0\ \forall\ \mu\neq z_{x_0}$, hence $T'_{x_0}(u)\leqslant C(u)\ \forall\ \mu\in D$, i. e. $T'_{x_0}\subseteq C$, Since T': $D\to F(D)$ is F-Continuous, therefore there exists a neighbourhood V_{x_0} of x_0 such that $T'[V_{x_0}]\subseteq D\cap C$, If $x\in V_{x_0}$, by $0<T'_x(z_x)\leqslant\sup_{\omega\in D}\{T'_\omega(z_x)\cdot V_{x_0}(\omega)\}=T'[V_{x_0}](z_x)\leqslant C(z_x)$, we have $C(z_x)=1$, i. e. $f(x)=z_x\in C$, which implies that if open set C such that $f(x_0)\in C$ then there exists V_{x_0} such that $f(x)\in C\ \forall\ x\in V_{x_0}$, therefore $f: D\to D$ is continuous at x_0, since $f: D\to D$ is continuous at each point of D, hence $f: D\to D$ is continuous.

D is H-Convex and compact subset of X, by lemma 1. 15, D is uniformly locally H-Convex, hence $\forall\ \varepsilon>0$, $\exists\ \eta\in(0,\varepsilon)$ such that $B_\varepsilon(x)$ is H-Convex relative to $B_\eta(x)$, by $f: D\to D$ is continuous, there extists a neighbourhood $B_{\delta_x}(x)$ of x such that $f(B_{\delta_x}(x))\subset B_\eta(f(x))$, D is compact and $C\subset\bigcup_{x\in D} B_{\delta_x}(x)$, hence we have $\{x_1, \cdots, x_n\}\subset D$ such that $D\subset\bigcup_{i=1}^{n} B_{\delta_{x_i}}(x_i)$, for $\{B_{\delta_{x_i}}(x_i)\mid i=1, 2, \cdots, n\}$ there extists a continuous partition of unity $\{\beta_i(x)\mid i=1, 2, \cdots, n\}: D\to[0,1]$, Let $\triangle_{n-1}=\{e_1\cdots e_n\}$ is a standard $(n-1)$—simplex, we can define

$$g: D\to\triangle_{n-1},\ g(x)=\sum_{i=1}^{n}\beta_i(x)e_i, \forall\ x\in D,$$

easy to doduce that $g: D \to \triangle_{n-1}$ is continuous, moreover by lemma 1. 14 there exists continuous mapping $h: \triangle_{n-1} \to D$ such that

$$h(e_{i_1} \cdots e_{i_k}) \subset \Gamma_{(f(x_{i_1}), \cdots, f(x_{i_k}))}$$

where $\{i_1, \cdots, i_k\}$ is a nonempty subset of $\{1, 2, \cdots, n\}$.

Since mapping $goh: \triangle_{n-1} \to \triangle_{n-1}$ and continuous, by Brouwer's fixed point theorem there exists $e \in \triangle_{n-1}$ such that $goh(e) = e$, Let $h(e) = x_\varepsilon$, then $x_\varepsilon \in D$ and $x_\varepsilon = h(e) = hogoh(e) = hog(x_\varepsilon)$, i. e. x_ε is a fixed point of hog, let $I(x) = \{i \in \{1, 2, \cdots, n\} \mid \beta_i(x) > 0\}$ $\forall x \in D$, hence $hog(x) = h(\sum_{i=1}^{n} \beta_i(x) e_i) = h(\sum_{i \in I(x)} \beta_i(x) e_i) \subset \Gamma_{(f(x_i), i \in I(x))}$ moreover, $i \in I(x)$ with $\beta_i(x) > 0$, i. e. $x \in B_{\delta_{x_i}}(x_i)$, by difinition of $B_{\delta_x}(x)$ easy to deduce that $f(B_{\delta_{x_i}}(x_i)) \subset B_\eta(f(x_i))$, $\therefore f(x) \in f(B_{\delta_{x_i}}(x_i)) \subset B_\eta(f(x_i))$, $\forall i \in I(x)$ then $f(x_i) \in B_\eta(f(x))$, moreover $B_\varepsilon(f(x))$ is H-Convex relative to $B_\eta(f(x))$, hence we have $h(g(x)) \in \Gamma_{(f(x_i), i \in I)} \subset B_\varepsilon(f(x))$ then $\forall\ \varepsilon > 0\ \exists\ x_\varepsilon \in D$ such that $x_\varepsilon = h(g(x_\varepsilon)) \in B_\varepsilon(f(x_\varepsilon))$, Let $\varepsilon_n = \frac{1}{n}$ $(n = 1, 2, \cdots)$ $\exists\ x_n \in B_{\varepsilon_n}(f(x_n))$, i. e. $d(x_n, f(x_n)) < \varepsilon_n$, by D is compacty, there exists $\{x_{n_j}\} \subset \{x_n\}$ such that $\lim_{j \to \infty} x_{n_j} = x_0 \in D$, moreoven $f: D \to D$ is continuous, $\lim_{j \to \infty} f(x_{n_j}) = f(x_0)$, $d(x_0, f(x_0)) \leqslant d(x_0, x_{n_j}) + d(x_{n_j}, f(x_{n_j})) + d(f(x_{n_j}), f(x_0)))$, when $j \to \infty$ $d(x_0, f(x_0)) \leqslant 0$, $\therefore d(x_0, f(x_0)) = 0$, i. e. $x_0 = f(x_0)$, by difinition of f, $x_0 = f(x_0) = z_{x_0}$, i. e. $T_{x_0}(x_0) = T_{x_0}(z_{x_0}) = \max_{u \in D} T_{x_0}(u)$, Then x_0 is a fixed point of T.

COROLLARY 2. 1. 1 ([3]) Let (X, d) is a metric space, $(X, \{\Gamma_A\})$ is a locally H-Convex space, D is nonempty compact subset of X and D is H-Convex, Let $f: D \to D$ is continuous, Then there exists $x_0 \in D$, x_0 is a fixed point of f.

Proof. we can define fuzzy function $T: D \to F(D)$ as follows:

$$\forall x \in D \quad T_x(\mu) = \begin{cases} 1 & \mu = f(x) \\ \frac{1}{2} & \mu \neq f(x) \end{cases}, \quad \forall \mu \in D$$

by T we have:

$\forall x \in D\ T'_x(\mu) = \begin{cases} 1 & \mu = f(x) \\ 0 & \mu \neq f(x) \end{cases}$, $\forall \mu \in D$ since $f: D \to D$ is continuous, it is easy to deduce that $T': D \to F(D)$ is F-Continuous, by theorem 2. 1 there exists $x_0 \in D$ such that $T_{x_0}(x_0) = \max_{\mu \in D} T_{x_0}(\mu)$, i. e. $T_{x_0}(x_0) = 1$, $\therefore x_0 = f(x_0)$.

THEOREM 2. 2 Let X is a normed linear space, D is a compact convex subset of X, $T: D \to F(D)$ is a fuzzy function, and $T': D \to F(D)$ is F-Continuous, Then there exists $x_0 \in D$, x_0 is a fixed point of T.

Proof. for any finite subset $\{x_1, x_2, \cdots, x_n\} \subset X$, We can define $\Gamma_A = C_0\{x_1, x_2, \cdots, x_n\}$, Then $(X, \{\Gamma_A\})$ is a locally H-Convex space, D is locally H-Convex and compactly, by theorem 2. 1 there exists $x_0 \in D$ such that $T_{x_0}(x_0) = \max_{\mu \in D} T_{x_0}(\mu)$, x_0 is a fixed point of T.

COROLLARY 2. 2. 1 (Schauder) Let X is a normed linear space, D is a compact convex subset of X, f: D→D is continuous, Then there exits $x_0 \in D$, such that $f(x_0) = x_0$.

Proof. define fuzzy function T: D→F (D) as follows:

$$\forall x \in D \quad T_x(\mu) = \begin{cases} 1 & \mu = f(x) \\ \frac{1}{2} & \mu \neq f(x) \end{cases}, \quad \forall \mu \in D$$

Then $\forall x \in D \quad T'_x(\mu) = \begin{cases} 1 & \mu = f(x) \\ 0 & \mu \neq f(x) \end{cases}$, $\forall \mu \in D$ it is easy to deduce that T': D→F (D) is F-Continuous, hence conclusion follows from theorem 2. 2.

COROLLARY 2. 2. 2 ([4]) Let $X = R^n$, D is a compact convex subset of X, T: D→F (D) is a fuzzy fouction, and T': D→F (D) is F-Continuous, Then there exits $x_0 \in D$ such that $T_{x_0}(x_0) = \max_{\mu \in D} T_{x_0}(\mu)$.

Proof. conclusion follows from theorem 2. 2.

COROLLORY 2. 2. 3 (Brouwer) Let $X = R^n$, D is a compact convex subset of X, f: D→D is continuous, Then there exists $x_0 \in D$, x_0 is a fixed point of f.

Proof. The same that proof of corollary 2. 2. 1, define fuzzy function T: D→F (D) and T': D→F (D), conclusion follows from corollory 2. 2. 2.

THEOREM 2. 3 Let $(X, \{\Gamma_A\})$ is an H-Space, D is a nonempty compact subset of X, and H-Convex, T: D→F (D) is fuzzy mapping such that: (1) there exists a real function $\alpha(x)$: D→ (0, 1] such that $\forall x \in D$ $(Tx)_{\alpha(x)} \neq \Phi$, $(T_x)_{\alpha(x)}$ is H-Convex, $\forall y \in D$ there exists a open set $O_y \subset T_\alpha^{-1}(y) = \{x \in D \mid y \in (T_x)_{\alpha(x)}\}$ and $\bigcup_{y \in D} O_y = D$, Then there exists $x_0 \in D$ such that $T_{x_0}(x_0) \geq \alpha(x_0)$, (2) In particular, if $\bar{\alpha}(x) = \max_{\mu \in D} T_x(u)$: D→ (0, 1] satisfies condition (1), Then there exists $x_0 \in D$, x_0 is a fixed point of T.

Proof. We can define set-valued mapping T_α: $D \to 2^D$ as follows: $\forall x \in D$, $T_\alpha(x) = (T_x)_{\alpha(x)}$, $\forall x \in D$, $T_\alpha(x) \neq \Phi$ and $T_\alpha(x)$ is H-Convex. Since D is compact subset and $D = \bigcup_{y \in D} O_y$, where $O_y \subset T_\alpha^{-1}(y)$, therefore there exists a finite set $\{y_1, \cdots, y_n\} \subset D$ such that $D \subset \bigcup_{i=1}^{n} O_{y_i}$, and there exists a continuous partition of unity $\{\beta_i(x) \mid i = 1, 2, \cdots, n\}$: D→ [0, 1], Let $\triangle_{n-1} = \{e_1 \cdots e_n\}$ is a standard (n−1) —simplex, we can define mapping g: $D \to \triangle_{n-1}$ as follows: $g(x) = \sum_{i=1}^{n} \beta_i(x) e_i$, $\forall x \in D$, Then g: $D \to \triangle_{n-1}$ and continuous, by lemma 1. 14, there exists continuous mapping h: $\triangle_{n-1} \to \Gamma_{(y_1, \cdots, y_n)} \subset D$ such that $h(e_{i_1}, \cdots, e_{i_k}) \subset \Gamma(y_{i_1} \cdots y_{i_k}) \subset D$, where $\{e_{i_1}, \cdots, e_{i_k}\}$ is a subset of $\{e_1, \cdots, e_n\}$, Since mapping go h: $\triangle_{n-1} \to \triangle_{n-1}$ is continuous, therefore there exist fixet point e such that go h (e) = e, Let $x_0 = h(e)$, then ∵ hog (x_0) = hogo h (e) = h (e) = x_0, ∴ x_0 is a fixed point of hog, On the other land, Since hog $(x_0) = h(\sum_{i=1}^{n} \beta_i(x_0) e_i) \in \Gamma_{(y_i, i \in I(x_0))}$, where $I(x_0) =$

$\{i\in\{1,2,\cdots,n\},\beta_i(x_0)>0\}$, therefore $\forall\ i\in I(x_0)$ from $\beta_i(x_0)>0$ we can obtain $x_0\in O_{y_i}\subset T_\alpha^{-1}(y_i)$, i. e. $y_i\in T_\alpha(x_0)\ \forall\ i\in I(x_0)$, moreover $T_\alpha(x_0)=(T_{x_0})_{\alpha(x_0)}$ is H-Convex, hence we have $\Gamma_{\{y_i, i\in I(x_0)\}}\subset T_\alpha(x_0)$, then $x_0=hog(x_0)\in T_\alpha(x_0)$, i. e. $T_{x_0}(x_0)\geqslant\alpha(x_0)$,, In particular, when $\bar{\alpha}(x)=\max_{\mu\in D}T_x(\mu)$ such that condition (1), $T_{x_0}(x_0)\geqslant\bar{\alpha}(x_0)=\max_{\mu\in D}T_{x_0}(\mu)$, then x_0 is a fixed point of T. this completes the proof.

COROLLARY 2. 3. 1 Let $(X,\{\Gamma_A\})$ is an H-Space, D is a nonempty compact subset of X, and H-Convex, $T: D\to 2^D$ is a set-valued mapping, If $\forall\ x\in D\ T(x)$ is nonempty and H-Convex, moreover $\forall\ y\in D$ there exists a open set $O_y\in T^{-1}(y)$, and $\bigcup_{y\in D}O_y=D$, Then there exists $x_0\in D$ such that $x_0\in T(x_0)$.

Proof. We can define mapping $\hat{T}: D\to F(D)$ as follows: $\forall\ x\in D\ \hat{T}_x(\mu)=\begin{cases}1 & \mu\in T(x)\\ 0 & \mu\notin T(x)\end{cases}\ \forall\ \mu\in D$, $\alpha(x)=1: D\to(0,1]$ the conclusion follows from theorem 2. 3 directly.

COROLLARY 2. 3. 2 ([3]) Let $(X,\{\Gamma_A\})$ is an H-Space, D is a nonempty compact H-Convex subset of X, $T: D\to 2^D$ is set-valued mapping such that $\forall\ x\in D$, $T(x)$ is nonempty and H-Convex, $T^{-1}(x)$ is open, Then there exists $x_0\in D$ such that $x_0\in T(x_0)$.

COROLLARY 2. 3. 3 (Browder) Let X is a Hausdorff topological vector space, $D\subset X$ is nonempty compact and convex, $T: D\to 2^D$ is a set-valued mapping such that $\forall\ x\in X$, $T(x)$ is nonempty convex and $T^{-1}(x)$ is open, Then there exists $x_0\in D$ such that x_0 is fixed point of T.

Proof. for any finite sub $A=\{x_1,\cdots,x_n\}\subset X$, We can define $\Gamma_A=Co\{x_1,\cdots,x_n\}$, then $(X,\{\Gamma_A\})$ is an H-Space, it is easy to deduce that satisfies conditions of corollary 2. 3. 2, therefore the conclusion follows from Corollary 2. 3. 2 directly.

REMARK 2. 3. 4 Corellary 2. 1. 1 is the theorem 1 of [3], Corellary 2. 2. 1 is the schauder's fixed poient theorem, corellary 2. 2. 2 is the theorem 2. 11 of [4], corellary 2. 2. 3 is the Brouwer fixed point theorem, they are all the special cases of theorem 2. 1, corellary 2. 3. 2 is the thorem 3 of [3], corellory 2. 3. 3 is the Browder's fixed point theorem, they are all the special cases of theorem 2. 3.

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