The Countable Additivity of Set-ValuedIntegrals and F-Valued Integrals

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Abstract: In this paper, integrals of Set-Valued functions and F-Valued functions are further investigated. At first, we show the countable additivity of Set-Valued integrals under two kinds of senses, then we extended these results to the circumstance of F-Valued integrals.

key words: additions, set-valued integral, F-Valued integral, countable additivity.

1. Introduction

Let $(X, \mathscr{A}, \mu)$ be a complate finite measure space, R^n be the n-dim Euclidean space. Let $\mathscr{P}_0(R^n)$, co K be the family of all nonempty subsets, nonempty compact convex subsets of R^n respectively. The countable addition on $\mathscr{P}_0(R^n)$ is defined as follow:

$$\sum_{n=1}^{\infty} B_n = \{\sum_{n=1}^{\infty} b_n : b_n \in B_n (n \geqslant 1), \sum_{n=1}^{\infty} \| b_n \| < \infty\}$$

for $B_n \in \mathscr{P}_0(R^n)$ $(n \geqslant 1)$, where $\| b_n \|$ is the Euclidean norm.

By using Kuratowski convergence, another addition can be defined as

$$\sum_{n=1}^{\infty} B_n = \underset{k}{Lim} \sum_{n=1}^{k} B_n$$

where $\{B_n\} \subset \mathscr{P}_0(R^n)$.

Let $\mathscr{F}(R^n)$ be the family of all fuzzy subsets on R^n, an element $\tilde{a} \in \mathscr{F}(R^n)$ is said to be a fuzzy number iff $\{r \in R^n : a(r) \geqslant \lambda\} \in \text{co } K$, for every $\lambda \in (0, 1]$, we use $\mathscr{F}^*$ to denote the set of all fuzzy numbers and further define the addition on $\mathscr{F}^*$ as:

$$(\sum_{n=1}^{\infty} \tilde{r}_n)(u) = sup\{\bigwedge_{n=1}^{\infty} \tilde{r}_n(u_n) ; u = \sum_{n=1}^{\infty} u_n, \sum_{n=1}^{\infty} \| u_n \| < \infty\}$$

where $\{\tilde{r}_n\} \subset \mathscr{F}^*$, let $\tilde{r}_1, \tilde{r}_2 \in \mathscr{F}^*$, we define

$$d(\tilde{r}_1, \tilde{r}_2) = \sup\{d(r_{1\lambda}, r_{2\lambda}) : \lambda \in (0, 1]\}$$

where $d(*, *)$ is the Hausdarff metric on $\mathscr{P}_0(R^n)$. Under the metric convergence for $\{\tilde{r}_n\} \subset \mathscr{F}^*$, we define another addition

$$\sum_{n=1}^{\infty} \tilde{r}_n = \underset{n}{Lim} \sum_{k=1}^{n} \tilde{r}_K$$

A set-valued function F: $X \to \mathscr{P}_0(R^n)$ is measurable if its graph is measurable, i. e.

$$G_r F = \{(x,y) : x \in X, y \in F(x)\} \in \mathscr{A} * \mathscr{B}(R^n).$$

A F-set-valued function is measurable if its λ—cut set-valued function $F_\lambda(x)$ is measurable for all $\lambda \in (0, 1]$, where $F_\lambda(x) = (\tilde{F}(x))_\lambda$ is the λ—cut set-valued function. The integral of $F: X \to \mathscr{P}_0(R^n)$ on a set $A \in \mathscr{A}$ is defined as:

$$\int_A F d\mu = \{\int_A f d\mu ; f \in S(F)\}$$

where $S(F) = \{f: f$ is an integrable selection of $F\}$.

The integral of $\tilde{F}: X \to F(R^n)$ on $A \in \mathscr{A}$ is defined as:

$$(\int_A \tilde{F} d\mu)(u) = sup\{\lambda \in (0,1]; u \in \int_A F_\lambda d\mu\}$$

The purpose of the paper is to prove that the integral of set-valued function has the property of countable additivity and further to show the extended corresponding result on F-valued integral.

2. On set-valued integrals

Theorem 1. Let $F_i: X \to \mathscr{P}_0(R^n)$ $(i \geqslant 1)$ be a sequence of measurable set-valued functions. Let $\psi_i: X \to [0, \infty)$ $(i \geqslant 1)$ be a sequence of measurable functions and assume that

i) $\| F_i(x) \| \leqslant \psi_i(x)$ for all $x \in X$,

ii) $\psi(x) = \sum_{i=1}^{\infty} \psi_i(x)$ is integrable;

Then

i) Set-valued function $F: X \to \mathscr{P}_0(R^n)$, $F(x) = \sum_{i=1}^{\infty} F_i(x)$ is integrable bounded;

ii) $S(F) = \sum_{i=1}^{\infty} S(F_i)$

iii) $\int_x F d\mu = \sum_{i=1}^{\infty} \int_x F_i d\mu$

Proof: It is clear that $\| F(x) \| \leqslant \sum_{i=1}^{\infty} \| F_i(x) \| \leqslant \sum_{i=1}^{\infty} \Psi_i(x) = \psi(x)$ then i) is proved.

Obverously. $S(F) \supset \sum_{i=1}^{\infty} S(F_i)$, hence to prove ii) it is sufficiently to verify that

$S(F) \subset \sum_{i=1}^{\infty} S(F_i)$.

For this aim, let $f \in S(F)$, we need to find a sequence $f_i: X \to R^n$ $(i \geqslant 1)$ such that

$f_i \in S\ (F_i)$, $f\ (x) = \sum_{i=1}^{\infty} f_i\ (x)$. Now, we define a set-valued function

$x \rightarrow G(x) = \{(u_1, u_2 \cdots\cdots); u_i \in F_i(x)(i \geqslant 1)\} \in I(R^n)$*(the Banach space of sequence*

$\{u_i\} \subset R^n$ *and the norm* $\| \{u_i\} \| = \sum_{i=1}^{\infty} \| u_i \| < \infty$)

By the measurability of F_i $(i \geqslant 1)$, it is easily that G is a measurable set-valued function (the definition of measurability is the same for a metric space). Next, let us define a continuous mapping to the following

$$H : I(R^n) \rightarrow X, H(\{u_i\}) = \sum_{i=1} u_i$$

consequently, the mapping

$$x \rightarrow H^{-1}(f(x)) = KerH + (f(x)/2^i)$$

is a measurable set-valued function from X to $I(R^n)$. since the intersection of two measurable set-valued function is also measurable and $f \in S\ (F)$, then the set-valued function $G\ (\cdot) \cap H^{-1}\ (\cdot)$ is nonempty measurable, therefore there exists an a . e. measurable selection

$$x \rightarrow \{f_i(x)\}, f_i(x) \in F_i(x), s.t. f(x) = \sum_{i=1}^{\infty} F_i(x) \quad a.e. x \in X$$

consequently. (ii) is proved.

(iii) is a direct result of (ii) (Q. E. D.)

Corollary 1. Let F_1, F_2 be integrably bounded set-valued function, then for a. $b \in R$

$$\int_X (aF_1 + bF_2) d\mu = a \cdot \int_X F_1 d\mu + b \int_X F_2 d\mu$$

Theorem 2. Let F_i: $X \rightarrow \mathscr{P}_0\ (R^n)$ $(i \geqslant 1)$ be a sequence of measurable set-valued functions, by the same conditions assumed as theorem 1. Then

$$\int_X (\sum_{i=1}^{\infty} F_i) d\mu = \sum_{i=1}^{\infty} \int_X F_i d\mu.$$

The proof can be easily obtained by using Corollary 1 and the dominated convergence theorem.

3. On F-valued integrals

Lemma 1. Let $\{\tilde{a}_n\} \subset \mathscr{F}^*$ be a sequence of F-numbers, $\sum_{i=1}^{\infty} \tilde{a}_n \in \mathscr{F}^*$, then

$$(\sum_{n=1}^{\infty} \tilde{a}_n)_\lambda = \sum_{n=1}^{\infty} a_{n\lambda}. \qquad \lambda \in (0,1]$$

Lemma 2. Let $\tilde{F}_n$ $(n \geqslant 1)$ be a sequence of measurable F-valued functions, ψ_n: $X \rightarrow [0, \infty)$ $(n \geqslant 1)$ be a sequence of measurable functions, further assume that

(i) $\sup_{\lambda \in (0,1]} \| F_{n\lambda}\ (x)\ \| \leqslant \psi_n\ (x)$, $(x \in X)$

(ii) $\psi(x) = \sum_{i=1}^{\infty} \psi_n(x)$ is integrable.

The mapping $\tilde{F}: X \to \mathscr{F}^*$, $x \to \sum_{i=1}^{\infty} \tilde{F}_n(x)$ is an integrably bounded F-valued function;

Proof: It is easy to see that $\tilde{F}$ is a F-valued function for $\lambda \in (0, 1]$, by Lemma 1 we have

$$F_\lambda(x) = \sum_{n=1}^{\infty} F_{n\lambda}(x),$$

then

$$\begin{aligned} sup_{\lambda \in (0,1]} \|F_\lambda(x)\| &\leqslant \sum_{n=1}^{\infty} sup_{\lambda \in (0,1]} \|F_{n\lambda}\| \\ &\leqslant \sum_{n=1}^{\infty} \psi_n(x) = \psi(x). \end{aligned}$$

(*Q. E. D*)

For $a \in R$, $\tilde{b} \in \mathscr{F}^*$, define $a \cdot \tilde{b}$ as $(a \cdot \tilde{b})_\lambda = a \cdot b_\lambda$ $(\lambda \in (0, 1])$

Corollary 2. Let $\tilde{F}_1$, $\tilde{F}_2$: $X \to R^n$ be integrably bounded F-valued functions, a, $b \in R$, then

$$\int_X (a \cdot \tilde{F}_1 + b\tilde{F}_2) d\mu = a \cdot \int_X \tilde{F}_1 d\mu + b \cdot \int_X \tilde{F}_2 d\mu$$

Theorem 4. Let $\tilde{F}_n$ $(n \geqslant 1)$ and $\tilde{F}$ be measurable F-valued functions, $\tilde{F} = \sum_{n=1}^{\infty} \tilde{F}_n$. Further assume that the conditions (i) and (ii) in Theorem 3 are satisfied, then

$$\int_X (\sum_{i=1}^{\infty} \tilde{F}_n) d\mu = \sum_{i=1}^{\infty} \int_X \tilde{F}_n d\mu.$$

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