A note on fuzzy Archimedean ordering

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Abstract

The notion of fuzzy Archimedean ordered fuzzy subgroup is introduced. It is proved that every fuzzy Archimedean ordered fuzzy subgroup with finite number of values is fuzzy order isomorphic to a fuzzy subgroup of the additive group of real numbers with its natural ordering.

Key words: *Fuzzy algebra, fuzzy subgroup, fuzzy partial order relation, fuzzy partially ordered fuzzy subgroup.*

1 Introduction

Fuzzy partially ordered fuzzy subgroup was defined and some of its fundamental properties were obtained in [1]. The object of the present note is to introduce the notion of **fuzzy Archimedean ordering** and to establish the validity of some results in the case of fuzzy Archimedean ordered fuzzy subgroups analogous to those obtained for fuzzy partially ordered fuzzy subgroups. Also Hõlder's theorem is used to prove that every fuzzy Archimedean ordered fuzzy subgroup with finite number of values is fuzzy order isomorphic to a fuzzy subgroup of the additive group of all real numbers with its natural ordering.

Unless otherwise mentioned the notions and notations are same as in [1].

2 Fuzzy Archimedean ordered fuzzy subgroups

Let G be a group with e as the identity element.
Let N denote the set of all positive integers.

Definition 2.1: *A fuzzy partially ordered fuzzy subgroup (λ, R) of G is said to be* **fuzzy Archimedean ordered** *if for all $a, b \in G$ such that $a \neq e, b \neq e$, there exist $n \in N$ satisfying $a^n b^{-1} \neq e$ and*

$$R(a^n b^{-1}, e) \geq M\left(R(a, e), R(b, e)\right).$$

Theorem 2.2: *For any fuzzy subset λ of G and a fuzzy relation R on λ, (λ, R) is a fuzzy Archimedean ordered fuzzy subgroup of G if and only if (λ_t, R_t) is an Archimedean ordered subgroup of G for all $t \in (0, 1]$.*

Proof: (λ_t, R_t) is a partially ordered subgroup of G (by Theorem 3.11 [1]). We now show that (λ_t, R_t) is Archimedean ordered . Let $t \in (0,1]$. Let $a, b \in$ G be such that $a \neq e, b \neq$ e and $(a, e), (b, e) \in R_t$. Then $R(a, e), R(b, e) \geq t$. Since (λ, R) is a fuzzy Archimedean ordered fuzzy subgroup of G, there exists $n \in N$ such that $a^n b^{-1} \neq e$ and

$$R(a^n b^{-1}, e) \geq M(R(a, e), R(b, e)) \geq t,$$

i.e, $(a^n b^{-1}, e) \in R_t$. So (λ_t, R_t) is Archimedean ordered.
Conversely, let (λ_t, R_t) be an Archimedean ordered subgroup of G for all $t \in (0, 1]$. Then λ is a fuzzy subgroup of G and R is a fuzzy partial order relation on λ. Let $x, y, t, v \in$ G and $R(x, y) > 0, R(t, v) > 0$. Let $m = M(R(x, y), R(t, v))$. Then $(x, y), (t, v) \in R_m$. Since (λ_m, R_m) is a partially ordered subgroup of G, $(xt, yv) \in R_m$ and thus

$$R(xt, yv) \geq M((R(x, y), R(t, v)).$$

So (λ, R) is a fuzzy partially ordered fuzzy subgroup of G. Let $a, b \in$ G be such that $a \neq$ e, $b \neq$ e. If possible, let for all

$$n \in N, \quad \text{such that} \quad a^n b^{-1} \neq e,$$
$$R(a^n b^{-1}) < M((R(a,e), R(b,e)).$$

Choose t such that $R(a^n b^{-1}, e) < t < M(R(a,e), R(b,e))$. Then $a, b \in \lambda_t$, $a \neq$ e, $b \neq$ e and $(a,e), (b,e) \in R_t$. Since (λ_t, R_t) is Archimedean ordered, there exists $n \in N$ such that $a^n b^{-1} \neq$ e and $(a^n b^{-1}, e) \in R_t$. i,e. $R(a^n b^{-1}, e) > t$, a contradiction. So there exists $n \in N$ such that $a^n b^{-1} \neq$ e and $R(a^n b^{-1}, e) \geq M(R(a,e), R(b,e))$. Therefore (λ, R) is a fuzzy Archimedean ordered fuzzy subgroup of G.

Theorem 2.3: *Let $(H_1, R_1) \subseteq (H_2, R_2) \subseteq \ldots \subseteq (H_m, R_m)$ be a chain of Archimedean ordered subgroups of G where $H_m = G$. Then there exists a fuzzy Archimedaen ordered fuzzy subgroup of G whose level subgroups are precisely the members of the chain.*

Proof: Let (λ, R) be the fuzzy partially ordered fuzzy subgroup as defined in Theorem 3.14 [1]. We now show that (λ, R) is fuzzy Archimedean ordered. Let $a, b \in$ G and $a \neq$ e, $b \neq$ e. Let k be the smallest positive integer such that $(a,e), (b,e) \in R_k$. Then $a, b \in H_k$. Since (H_k, R_k) is Archimedean ordered, there exists $n \in N$ such that
$a^n b^{-1} \neq$ e and $(a^n b^{-1}, e) \in R_k$ and hence

$$R(a^n b^{-1}, e) \geq M(R(a,e), R(b,e)).$$

So (λ, R) is a fuzzy Archimedean ordered fuzzy subgroup of G.

Theorem 2.4: *Let H be a group with $\bar{e}$ as the identity element and $f : G \to H$, a monomorphism.*
(i) Let $(\mu, \bar{R})$ be a fuzzy Archimedean ordered fuzzy subgroup of H and $R : G \times G \to [0,1]$ be defined by

$$R(x,y) = \bar{R}(f(x), f(y))$$

for all $(x,y) \in G \times G$. *Then* $(f^{-1}(\mu), R)$ *is a fuzzy Archimedean ordered fuzzy subgroup of G.*

(ii) Let (λ, R) *be a fuzzy Archimedean ordered fuzzy subgroup of G where* λ *has the "sup property".*

If $\bar{R} : f(G) \times f(H) \to [0,1]$ *is defined by*

$$\bar{R}(f(x), f(y)) = R(x,y)$$

for all $(f(x), f(y)) \in f(G) \times f(G)$.

Then $(f(\lambda), \bar{R})$ *is a fuzzy Archimedean ordered fuzzy subgroup of f(G).*

Proof: (i) $(f^{-1}(\mu), R)$ is a fuzzy partially ordered fuzzy subgroup of G (by Theorem 3.15 [1]). Let $a, b \in$ G be such that $a \neq$ e and $b \neq$ e. Let $f(a) = x$ and $f(b) = y$. Then $x \neq \bar{e} \neq y$, since f is a monomorphism. Since $(\mu, \bar{R})$ is fuzzy Archimedean ordered there exists $n \in N$ such that $x^n y^{-1} \neq \bar{e}$ and $\bar{R}(x^n y^{-1}, \bar{e}) \geq M(\bar{R}(x, \bar{e}), \bar{R}(y, \bar{e}))$ and hence $R(a^n b^{-1}, e) \geq M(R(a, e), R(b, e))$.
Also since f is a monomorphism, $a^n b^{-1} \neq e$. So $(f^{-1}(\mu), R)$ is a fuzzy Archimedean ordered fuzzy subgroup of G.
(ii) The proof is similar to (i).

Definition 2.5: *Let* λ *and* μ *be two fuzzy subgroups of G and H respectively.* λ *is said to be* **fuzzy isomorphic** to μ if there exists an isomorphism $f : \lambda_0 \to \mu_0$ such that $\mu(y) = \lambda(x)$ where $f(x) = y$ for all $y \in \mu_0$. Let (λ, R) and $(\mu, \bar{R})$ be fuzzy partially ordered fuzzy subgroups of G and H respectively. Then (λ, R) is said to be *fuzzy order isomorphic* to $(\mu, \bar{R})$ if λ is fuzzy isomorphic to μ and

$$R(x,y) = \bar{R}(f(x), f(y)) \quad \text{for all} \quad (x,y) \in \lambda_0 \times \lambda_0.$$

Lemma 2.6: *Let* (H_i, R_i) *be an Archimedean ordered* *subgroup of G for $i = 1, 2$ such that $H_1 \subseteq H_2$ and $R_1 \subseteq R_2$. By Hölder's theorem (H_i, R_i) is isomorphic to (K_i, S_i) where K_i is a subgroup of the additive group of all real numbers and*

$$S_i = \{(a, b) \in K_i \times K_i \quad ; a \geq b\}$$

for $i = 1, 2$. If f_i is the order isomorphism of (H_i, R_i) onto (K_i, S_i), then $f_2|H_1 = f_1$.

Theorem 2.7: *Every fuzzy Archimedean ordered fuzzy subgroup with finite number of values is fuzzy order isomorphic to a fuzzy sugroup of the additive group of all real numbers with its natural ordering.*

Proof: Let (λ, R) be a fuzzy Archimedean ordered fuzzy subgroup of G with non-zero values $t_1, t_2, \ldots, t_m$ where $t_1 > t_2 > \ldots > t_m$. Let

$$H_i = \lambda_{t_i}, \quad R_i = R_{t_i} \quad \text{for} \quad i = 1, 2, \ldots, m.$$

Then

$$(H_1, R_1) \subseteq (H_2, R_2) \subseteq \cdots \subseteq (H_m, R_m) \qquad (1).$$

is a chain of Archimedean ordered subgroups of G. By Hölder's theorem there exist an order isomorphism f_i of (H_i, R_i) onto (K_i, S_i) where K_i is a subgroup of the additive group of all real numbers and

$$S_i = \{(a, b) \in K_i \times K_i \quad ; a \geq b\}$$

for $i = 1, 2, \ldots, m$ By Lemma 2.6 $f_m|H_i = f_i$ for $i = 1, 2, \ldots, m-1$. Now

$$(K_1, S_1) \subseteq (K_2, S_2) \subseteq \cdots \subseteq (K_m, S_m) \qquad (2).$$

is a chain of Archimedean ordered subgroups of the additive groups of all real number with its natural ordering. Let (μ, R') be the fuzzy Archimedean ordered fuzzy subgroup of the additive group of all real numbers determined by the chain (2) and numbers $t_1, \ldots, t_m$. We note that

$$\lambda_0 = H_m, \quad \mu_0 = K_m$$

and $f_m : H_m \to K_m$ is a group homomorphism. Also it follows from the construction of (μ, R') that $\mu(f(x)) = \lambda(x)$ and

$$R'(f_m(x), f_m(y)) = R(x,y) \quad \forall x, y \in H_m.$$

So f_m is an order isomorphism of (λ, R) onto (μ, R).

Theorem 2.8: *Let (λ, R) and (μ, S) be two fuzzy Archimedean ordered fuzzy subgroups of two groups G and H (with $\bar{e}$ as the identity element of H) respectively. Then $(\lambda \times \mu, R \times S)$ is a fuzzy Archimedean ordered fuzzy subgroup of $G \times H$ where*

$$(R \times S)((x,y),(u,v)) = M((R(x,u), S(y,v))$$

for all $x, u \in G$ and $y, v \in H$.

Proof: $(\lambda \times \mu, R \times S)$ is a fuzzy partially ordered fuzzy subgroup of G× H (By Theorem 3.17 [1]). We now show that $(\lambda \times \mu, R \times S)$ is fuzzy Archimedean ordered. Let $(x,y),(u,v) \in G \times H$ and $(x,y) \neq (e,\bar{e})$ and $(u,v) \neq (e,\bar{e})$. Then since (λ, R) and (μ, S) are fuzzy Archimedean ordered there exists $n, m \in N$ such that

$$\begin{gathered} x^n u^{-1} \neq e, y^n v^{-1} \neq \bar{e} \qquad \text{and} \\ R(x^n u^{-1}, e) \geq M(R(x,e), R(u,e)), \\ S(y^m v^{-1}, \bar{e}) \geq M(S(y,\bar{e}), S(v,\bar{e})). \end{gathered}$$

Let $n = max\{n, m\}$. Now

$$\begin{gathered} (x,y)^n(u,v)^{-1} = (x^n u^{-1}, y^n v^{-1}) \neq (e,\bar{e}). \quad \text{Also} \\ (R \times S)(((x,y)^n(u,v)^{-1}),(e,\bar{e})) \\ = M(R(x^n u^{-1}, e), S(y^n v^{-1}, \bar{e})) \\ \geq M(R(x,e), R(u,e), S(y^n v^{-1}, \bar{e})). \end{gathered}$$

Again

$$\begin{aligned} S(y^n v^{-1}, \bar{e}) &= S(y^{n-m}(y^m v^{-1}), \bar{e}^{n-m}\bar{e}) \\ &\geq M(S(y^{n-m}, \bar{e}^{n-m}), S(y^m v^{-1}, \bar{e})) \\ &\geq M(S(y, \bar{e}), S(y, \bar{e}), S(v, \bar{e})) \\ &= M(S(y, \bar{e}), S(v, \bar{e})). \end{aligned} \tag{3}$$

So from (3)

$$\begin{aligned} &(R \times S)(((x, y)^n (u, v)^{-1}), (e, \bar{e})) \\ &\geq M(M(R(x, e), S(y, \bar{e}), M((R(u, e), S(v, \bar{e}))) \\ &= M((R \times S)((x, y), (e, \bar{e})), (R \times S)((u, v), (e, \bar{e}))). \end{aligned}$$

So $(\lambda \times \mu, R \times S)$ is fuzzy Archimedean ordered.

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