Explicit formulae of two input fuzzy control

J. Varga and L. T. Kóczy

Dept. of Telecommunication and Telematics
Technical University of Budapest
Stoczek u. 2, Budapest, H-1111 Hungary
Fax: +36-(1)-463-3107

1 Introduction

At present great attention concentrated on fuzzy logic in the industrial world. Fuzzy control is one of the most popular application areas of fuzzy logic. Since this time more than 2000 industrial application consist fuzzy control.

Since the beginning of the 90th the articles of the fuzzy literature only dealt the linguistic rule based, heuristic fuzzy controllers. That time began the examination of the explicit formulas of fuzzy controllers. We can answer questions — like which type of transfer functions can be realized with fuzzy controllers, or how appropriate can be the given function approximated with fuzzy controllers –/ by examining the explicit output formulas.

In the early 90th more author had proved that the fuzzy controllers are *universal approximators*. Kosko had showed that an additive fuzzy system uniformly approximates a given $f: x \to y$ if X is compact and f is continuous [6]. At the same time Wang has proved that for any f real continuous function on a compact set exists a fuzzy control function that approximate f with arbitrary accuracy. Nguyen and Kreinovich generalized Wang's theorem for any t- and s-norm and arbitrary averaging type defuzzification operator. The common characteristic feature of the last three theorems is the *very dense rule base and unbounded support.* In the practical applications both the rule base and the support are bounded.

Since this time the fuzzy literature investigated almost the one-input fuzzy controllers. In this article we take our interest in two-input controllers.

2 One input formulae and approximation

Our first question is what crisp formula is the equivalent of a given fuzzy controller. El Hajjaji and Rachid [3] have given a partial answer to this question for a special one-input fuzzy control system. In their model both antecedents and consequents are identical, isosceles triangular. They examined a Mamdani [8] controller with a

Center of Area defuzzification method, and obtained the following explicit formula for the output of the controller:

$$y^* = (i + \frac{1}{2})b + \frac{(d_2 - 1)^2 - (d_1 - 1)^2}{2d_1 - d_1^2 + d_2} \times \frac{b}{2} \tag{1}$$

where b is the common base for the antecedents and consequent, and (d_1, d_2) is the fuzzified version of observation x^*.

In their paper Kóczy and Sugeno [4] have derived the formula for the same controller however using Center of Gravity defuzzification, instead of the COA method:

$$y^* = (i + \frac{1}{2})b + \frac{(d_1 - 1)^2 - (d_2 - 1)^2}{2d_1 - d_1^2 + 2d_2 - d_2^2} \times \frac{b}{2} \tag{2}$$

the notations are the same as used in (1).

They showed, that the maximal deviation between (1) and (2) is 2% of the common base length. In [4] explicit output formulae for the same rule based Larsen-style controller [7] have been given as well, both for the COA and COG defuzzification method. The output formula for the Larsen-style controller with COG defuzzification is more interesting than the previous results:

$$y^{*'} = (i + \frac{1}{2})b + (1 - 2d_1) \times \frac{b}{2}, \tag{3}$$

as (3) is a linear function of x^* (as d_1 is linear itself).

The analogue maximum deviation value for these Larsen-style controllers is 6 % of the common base length. The two defuzzification methods do not deliver very much different conclusions, so they restrict investigations to the use of the COG defuzzification technique in the next.

The Mamdani-controller general trapezoidal antecedents and consequents has the following structure:

$$y^{*'} = C_7 + C_8 x^* + \frac{C_9 + x^8}{C_4' + C_5' x^* + C_6' x^{*2}}. \tag{4}$$

The output formula is given in [4] for the general Takagi-Sugeno controller [11] as well. The structure of the controller's output function is

$$y^* = c_1 x^{*^2} + c_2 x^* + c_3 + \frac{c_4}{c_5 + c_6 x^*} \tag{5}$$

(5) has a different behaviour from the Mamdani- or Larsen-controllers as it has a parabolic member.

The Sugeno-controller is a fuzzy controller with crisp singleton consequents. This controller is a special case of both the Mamdani-controller and the Takagi-Sugeno controller. It has been shown, that by using adaptive controllers of this type, it is possible to construct stable fuzzy controllers for non-linear systems [5]. Also it has been shown that the Sugeno-controller is suitable for the generation of arbitrary continuous functions [1].

In [4] an exact output function is calculated, but here we give only the structural form :

$$y^* = c_1 + \frac{c_2}{c_3 x^* + c_4} \tag{6}$$

If the antecedents are equidistant isosceles triangulars, then (6) is identical with (3). So *The isosceles triangular Larsen-controller with COG defuzzification is identical in its behaviour with the Sugeno-controller.*

In the next sections we will calculate the analogue formulae for two-input controllers.

3 Two input formulae with a single rule

In this section we examine a special fuzzy controller that has only one rule. The method we use here will easily be generalized to multi-rule controllers.

Suppose a two input (x_1^* , x_2^*) fuzzy controller has only one (R_1) rule, and (R_1) has the following format:

$$\textbf{If}\, X_1\, is\, A_{11}\, and\, X_2\, is\, A_{22},\, \textbf{then}\, Y\, is\, Q_{11}\,. \tag{7}$$

We have to restrict the shapes of both the antecedents and the consequents. In industrial fuzzy controllers the shape of all terms in the rules is trapezoidal, so in this paper we always suppose this shape. It is reasonable to choose the consequent shapes so that they have a single core point, however, in other applications, in linguistic fuzzy reasoning, also trapezoidal consequents might be meaningful.

We will only consider the COG method of defuzzification, because there is not too much deviation between the COG and COA methods, as shown in [4], and calculation by COG method is simpler. We used **Maple** for some of our calculations.

The trapezoidal A_{ij} consequent has four characteristic points (the min of support, the min of the core, the max. of the core, the max. of the support), a_{ij}, b_{ij}, c_{ij} and d_{ij}, where $a_{ij} \leq b_{ij} \leq c_{ij} \leq d_{ij}$.

The observation (x_1^*, x_2^*) gives the degrees of matching $\mu_{A_{11}}(x_1) = D_1$ and $\mu_{A_{21}}(x_2) = D_2$ for the antecedents A_{11} and A_{21}. From these we can calculate the output of the Mamdani-controller with applying the COG defuzzification.

We concentrate on the non-trivial subregion

$$(a_{11} \leq x_1 \leq b_{11},\; a_{21} \leq x_2 \leq b_{21}) \tag{8}$$

Outside of the (8) D_1 or D_2 are 0. In the regions ($b_{11} \leq x_1 \leq c_{11}$ or $b_{21} \leq x_2 \leq c_{21}$) D_1 or D_2 is 1. We chose one of the remaining four subregions, examining the other three regions, we gave analogue output functions.

The membership degrees D_1 and D_2 can be expressed:

$$D_1 = \frac{a_{11} - x_1}{a_{11} - b_{11}} \quad and\, D_2 = \frac{a_{21} - x_2}{a_{21} - b_{21}}$$

3.1 The min aggregating operator

From application points of view there can be two T-norm used in (7) to aggregate the D_1 and D_2 degrees of matching of the inputs x_1^* and x_2^*, the min function and the Algebraic T-norm. The remaining T-norms have bigger computational complexity.

Applying the min function as a T-norm, for our two dimensional input space, the rectangular region can be devided into two parts on one part $\min\{D_1, D_2\} = D_1$ and in the other $\min\{D_1, D_2\} = D_2$. The points where membership degrees d_{11} , d_{12} are equal, can be expressed by

$$(a_{11} - x_1)\,(a_{21} - b_{21}) = (a_{21} - x_2)\,(a_{11} - b_{11})\,. \tag{9}$$

(9) describes a straight line, that is the diagonal of the (8) rectangular region. For the observations (x_1^*, x_2^*) where

$$x_2 \leq \frac{(x_1 - a_{11})\,(a_{21} - b_{21})}{(a_{11} - b_{11})} + a_{21}\,,$$

(x_1^*, x_2^*) is 'below the line $D_1 \leq D_2$ and $D_{11} = \min D_1, D_2 = D_1$. For the other points 'above the line', $D_{11} = D_{11} = \min D_1, D_2 = D_2$.

The aggregated D_{11} degree of matching allows us to calculate the output function of the Mamdani-controller.

The four characteristic point of the Q_{11} consequent are p, q, r, s , where $p \leq q \leq r \leq s$. The center of this trapezoid is

$$y_{trap} = \frac{(q-p)^2 - (s-r)^2}{3\,(q-p+s-r)} D - \frac{p\,(p+2r-q) - s\,(s+2q-r)}{3\,(q-p+s-r)} - \tag{10}$$

$$\frac{p^2(s-p+q+r) - 2\,ps(r+q) + s^2(p-s+q+r)}{(-3\,s+3\,r-3\,q+3\,p)\,(2\,s+D(r-s+p-q)-2\,p)}\,.$$

where D_{11} is the degree of matching, for this antecedent. Let us substitute the value of $\min\{D_1, D_2\}$ into (10), then we give the explicit output value of the controller, for the points 'below' the line:

$$y_{trap} = c_1\, x_1 + c_2 + \frac{c_4}{c_5\, x_1 + c_6}\,. \tag{11}$$

We gave the same structure for the points 'above' the line with different c_m constants. We used this notation with constants c_m in order to make the structure clean, otherwise these constants are functions of the rule base and antecedent parameters

$$c_m = f(a_{11}, a_{21}, b_{11}, b_{21}, p, q, r, s)$$

The first two terms of (11) are linear the last term is hyperbolic. We could give no exact approximation for this non-linear term in general cases.

Using triangular consequents the center is simpler than (10):

$$y_{tria} = \frac{p+q+s}{3} + \frac{D_1\,(2\,q-s-p)}{3} + \frac{2\,q-s-p}{3\,D_1 - 6}\,. \tag{12}$$

However the output formula has the same structure as (10), for the points 'below the line':

$$y_{tria} = \frac{(s+p-2\,q)\,x_1}{3\,a_{11}-3\,b_{11}} - \frac{-3\,a_{11}q+b_{11}s+b_{11}q+b_{11}p}{3\,a_{11}-3\,b_{11}} + \tag{13}$$

$$\frac{(a_{11}-b_{11})(s+p-2q)}{3\,a_{11}-6\,b_{11}+3\,x_1}.$$

If the consequent shape is symmetrical, $q_{11}-p_{11}=s_{11}-r_{11}$ for trapezoidal case or $q_{11}-p_{11}=s_{11}-q_{11}$ for triangular case, then the output formulas would be crisp constants.

3.2 The algebraic t-norm aggregating operator

In the previous section we can give output formula for the subsets of the region of interest. If the algebraic t-norm is the aggregating operator, then there exists an explicit output formula for the whole observed range. The aggregated degree of matching can be expressed:

$$D_{11}(x_1,\,x_2) = \frac{(a_{11}-x_1)\,(d_{21}-x_2)}{(a_{11}-b_{11})\,(d_{21}-c_{21})}. \tag{14}$$

Substituting (24) to the center formula we gave the following structure for both the triangular- and the trapezoidal consequents:

$$y_{trap} = c_1\,x_1 + c_2\,x_2 + c_3\,x_1x_2 + c_4 + \frac{c_5}{c_6\,x_1 + c_7\,x_2 + c_8\,x_1x_2 + c_9}. \tag{15}$$

(15) has a bit complicated behaviour, than (11) or (13).

4 Two input formulae with four rules

In the previous section a simplified control model is discussed. We extend our investigation to general multi-rule control systems. The rules have the following format:

$$\textbf{If}\ X_1\ \text{is}\ A_{1i}\ \text{and}\ X_2\ \text{is}\ A_{2j}\ \textbf{then}\ Y\ \text{is}\ Q_{ij}\,. \tag{16}$$

Suppose that A_{11}, A_{12}, A_{21} and A_{22} give positive and nontrivial degrees of matching for $\mathbf{x}^* = (X_1^*,\, x_2^*)$, then the following four rules fire: R_{11}, R_{12}, R_{21} and R_{22}. We assumed that $0 < D_{ij} < 1$ so

$$c_{11} \le a_{12} \le x_1 \le d_{11} \le b_{12}$$

holds for x_1 and

$$c_{21} \le a_{22} \le x_2 \le d_{21} \le b_{22}$$

for x_2.

From here the D_{ij}^* aggregated degree of matching can be calculated. For example D_{21}^* is the following:

$$D_{21}^* = \mu_{A_{12}}(x_2) \wedge \mu_{A_{21}}(x_2) = D_{12} \wedge D_{21} = \frac{a_{12}-x_1}{a_{12}-b_{12}} \wedge \frac{d_{21}-x_2}{d_{21}-c_{21}},$$

where $\wedge$ is one of the t-norms, the min or the algebraic t-norm.

4.1 min as a t-norm

First we have to determine the regions, where an exact output formula can be given. The sections, where the (x_1, x_2) observations give equal membership degree for the two antecedents of a given rule can be calculated analogue with the one-rule controller.

E.g. for the A_{1i} and A_{2j} antecedents of the rule R_{ij} this section is

$$(a_{1i}, a_{2j}),\ (b_{1i}, b_{2j})\,.$$

There exist similar sections for all the four rules. One end point of these sections is the corner of the region of interest. As we know from the basic geometry n general line dived the space into $\frac{n(n+1)}{2}+1$ parts. In our case these four lines divide the observed region into maximum 11 subparts. In these subparts there exists an explicit crisp output formula of the fuzzy controller, with analogue the one-rule controller.

The COG defuzzfied output formula of the two-input fuzzy-controllers are the following:

$$y^*_{trap} = \frac{\sum_{i,j=1..2} A(D^*_{ij}, Q_{ij}) \cdot y(D^*_{ij}, Q_{ij})}{\sum_{i,j=1..2} A(D^*_{ij}, Q_{ij})}, \tag{17}$$

where y is the scaled or α-cut center of the Q_{ij} consequent, depends on the inference method. The other function, A is the scaled or α-cut area of Q_{ij}.

4.1.1 Mamdani-controller

For Mamdani-controllers y has already derived in section 3.1, the area of trapezoidal consequents is the following:

$$A_{trap}(D^*_{ij}, Q_{ij}) = (p_{ij} - q_{ij} + r_{ij} - s_{ij})\,\frac{{D^*_{ij}}^2}{2} + (s_{ij} - p_{ij})\,D^*_{ij}\,. \tag{18}$$

Applying (18) and (10) the COG defuzzyfied output of a Mamdani-controller for trapezoidal consequents has the following structure:

$$y^* = \frac{c_1 x_1^3 + c_2 x_2^3 + c_3 x_1^2 + c_4 x_2^2 + c_5 x_1 + c_6 x_2 + c_7}{c_8 x_1^2 + c_9 x_2^2 + c_{10} x_1 + c_{11} x_2 + c_{12}} \tag{19}$$

(19) is an explicit crisp output formula for two-input Mamdani-controllers. The c_m constants are differ from one sub-region to other, and they are not so simple. The output formula is the result of a polinom division. The denominator of (19) is a two-value parabolic polinom, the numerator is a two-value third degree polinom with different c_m coefficients, so in general there is no chance to simplify it. The coefficients are a functions of the rule parameters.

$$c_m = f(a_{kl}, b_{kl}, c_{kl}, d_{kl}, p_{kl}, q_{kl}, r_{kl}, s_{kl})_{k=1..2,l=1..2}$$

If the consequents have triangular shape, then the output has the same structure as (19).

4.1.2 Larsen- and Sugeno-controller

In the 3.1-th section we do not discuss the Larsen- and Sugeno-controllers, because for one-rule systems their output formula is a crisp constant.

For Larsen- and Sugeno-controller the $\widehat{y}(D^*_{ij}, Q_{ij})$ does not depend on the degree of matching only on the shape of the Q_{ij}:

$$\widehat{y}(D^*_{ij}, Q_{ij}) = \widehat{y}(Q_{ij}), \tag{20}$$

and $\widehat{A}(D^*_{ij}, Q_{ij})$ is linear function of D^*_{ij}, so (17) can be written by:

$$y^* = \frac{\sum_{i,j=1..2}\left(D^*_{ij}\cdot W_{ij}\right)\cdot Y_{ij}}{\sum_{i,j=1..2} D^*_{ij}\cdot W_{ij}}. \tag{21}$$

The W_{ij} constant is the weight of the Q_{ij} consequent, and the Y_{ij} crisp value is the center of Q_{ij}. The only difference between the Larsen- and the Sugeno-controller is the value of W_{ij}s. For Sugeno-controller

$$W_{ij} = 1. \tag{22}$$

The explicit output formula of these controllers has the structure:

$$y^* = \frac{c_1 x_1 + c_2 x_2 + c_3}{c_4 x_1 + c_5 x_2 + c_6}. \tag{23}$$

In figure 1 the whole input region of a general four-rule Mamdani-fuzzy control system with COG defuzzification can be seen. The antecedents and consequents are in table 4.1.2:

A_{11}	trapezoidal, with characteristic points (0.5, 2, 4, 6.7)
A_{12}	trapezoidal, with characteristic points (4.1, 7, 8, 8.5)
A_{21}	trapezoidal, with characteristic points (0.5, 2, 3, 5)
A_{22}	trapezoidal, with characteristic points (3.2, 5, 7, 8)
q_{11}	triangular, with characteristic points (1, 2, 5)
q_{12}	triangular, with characteristic points (6, 12, 13)
q_{21}	triangular, with characteristic points (3, 3.5, 6)
q_{22}	triangular, with characteristic points (2.5, 11, 11.5)

Table 1: The Antecedents and consequents of a controller

4.2 Algebraic t-norm

There is no need to divide the input region into subregions, if an algebraic t-norm is the aggregating operator, but the output formulas are complicated. In this case the D^*_{ij} aggregated degree of matching can be expressed by:

$$D_{11}(x_1, x_2) = \frac{(a_{11} - x_1)(d_{21} - x_2)}{(a_{11} - b_{11})(d_{21} - c_{21})}. \tag{24}$$

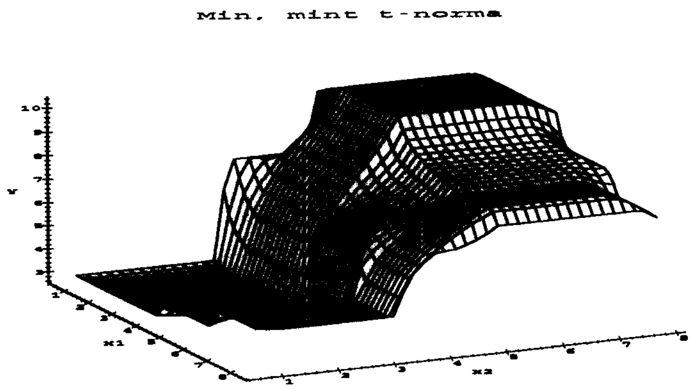

Figure 1: Output of a Mamdani-controller.

For Mamdani-controller the output formula can be derived by substituting (24) into (17), the result has the following structure:

$$\frac{\sum_{i=0..3,\,j=0..3} c_{1ij}\, x_1^i\, x_2^j}{\sum_{i=0..2,\,j=0..2} c_{2ij}\, x_1^i\, x_2^j} \tag{25}$$

For Larsen- and Sugeno-controller the output formula is much simpler:

$$y^* = \frac{c_1 x_1 + c_2 x_2 + c_3\, x_1 x_2 + c_4}{c_5 x_1 + c_6 x_2 + c_7\, x_1 x_2 + c_8}.$$

The output of the Larsen-controller using algebraic t-norm aggregating operator, COG defuzzification is in figure 2. The parameters can be seen in table 4.1.2.

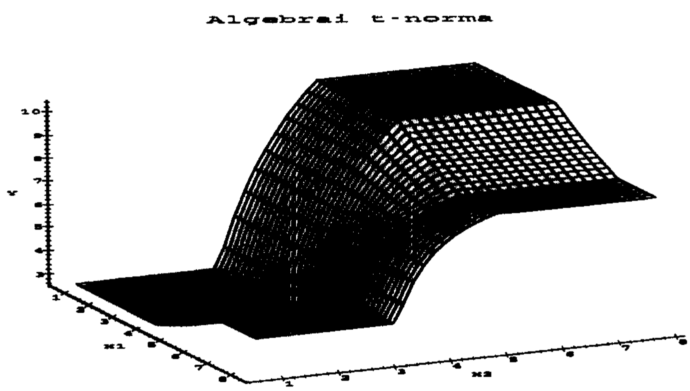

Figure 2: Output of a Larsen-controller.

5 Conclusion

In the previous sections we have shown, that it is possible to give the exact crisp output formulae for two input fuzzy controllers. We had showed that fuzzy controllers are not fuzzy systems in the sense that they can be substituted by crisp functions. This was

We pointed at differences between the aggregating operators min and the algebraic t-norm. By using the min we have to divide the firing region into parts, and can give explicit crisp output formulae for these regions.

With the algebraic t-norm, no division is , but the output formulae are not as simple as the output functions of the min. The output formulae are polynomial fraction, and in general cannot give a linear term.

The methods we used can be generalized to determine the output crisp function of an n-input fuzzy controllers.

References

[1] P. Bauer, E. P. Klement, B. Moser, A. Likeermoser: Modeling of control functions by a fuzzy controllers, *H. Nguyen, M. Sugeno, R. Tong, R. Yager: Theoretical aspects of fuzzy control*, J. Wiley, 1994, to appear.

[2] D. G. Burkhard and P. P. Bonissone Automated fuzzy knowledge base generation and tuning. *IEEE Int. Conf on Fuzzy Systems* , San Diego, 1992. 176-179.

[3] A. El Hajjaji and A. Rachid : Explicit formulas for fuzzy controller. FSS **62** (1994), pp. 135-141.

[4] L. T Kóczy and M. Sugeno: Explicit formulas for fuzzy control systems and the approximator property.

[5] J. T. K. Koo: Design of stable adaptive fuzzy control, M. Phil thesis, *Div. of Info. Engineering*, The Chinese University of Hong Kong, 1993.

[6] B. Kosko: Fuzzy systems as universal approximators, *IEEE Int. Conf on Fuzzy Systems*, San Diego, 1992, pp. 1153-1162.

[7] P. M. Larsen: Industrial application of fuzzy logic control, *Int. J. of Man Machine Studies* , **12** (1980), 3-10.

[8] E. H. Mamdani and S. Assilian: An experiment in linguistic synthesis with a fuzzy logic controller, *Int. J. Man-Machine Studies* , **7**, (1975), 1-13.

[9] H. T. Nguyen, V. Kreinovich: On approximations of controls by fuzzy systems, *TR 92-93/302*, LIFE Chair of Fuzzy Theory, TIT, Tokyo, 1992.

[10] M. Sugeno and T. Yasukawa: A fuzzy-logic-based approach to qualitative modeling. *IEEE Tr. FS 1*, (1993), 7-31.

[11] T. Takagi and M. Sugeno: Fuzzy identification of systems and its applications to modeling and control, IEEE Trans. on Sys. Man and Cybernetics, **15** (1985), 116-132.

[12] L. X. Wang: Fuzzy systems are universal approximators, *IEEE Int. Conf on Fuzzy Systems*, San Diego, 1992, pp. 1163-1169.